Chapter 2 – Sequences

Course Title: Real Analysis 1 **Course Code:** MTH321
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A sequence (of real numbers, of sets, of functions, of anything) is simply a list. There is a first element in the list, a second element, a third element, and so on continuing in an order forever. In mathematics a finite list is not called a sequence; a sequence must continue without interruption. Formally it is defined as follows:

Sequence

A sequence is a function whose domain of definition is the set of natural numbers.

Or it can also be defined as an ordered set.

Notation:

An infinite sequence is denoted as

$\{s_n\}_{n=1}^{\infty}$ or $\{s_n : n \in \mathbb{N}\}$ or $\{s_1, s_2, s_3, ...\}$ or simply as $\{s_n\}$ or by (x_n).

The values s_n are called the *terms* or the *elements* of the sequence $\{s_n\}$.

e.g. i) $\{n\} = \{1, 2, 3, \ldots\}$.

ii) $\left\{\dfrac{1}{n}\right\} = \left\{1, \dfrac{1}{2}, \dfrac{1}{3}, \ldots\right\}$.

iii) $\{(-1)^{n+1}\} = \{1, -1, 1, -1, \ldots\}$.

iv) $\{2, 3, 5, 7, 11, ...\}$, a sequence of positive prime numbers.

Subsequence

It is a sequence whose terms are contained in given sequence.

A subsequence of $\{s_n\}_{n=1}^{\infty}$ is usually written as $\{s_{n_k}\}^{\infty}$.

Increasing Sequence

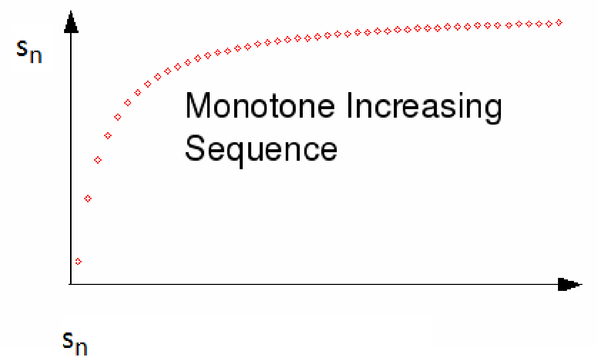
A sequence $\{s_n\}$ is said to be an increasing sequence if $s_{n+1} \geq s_n \quad \forall\ n \geq 1$.

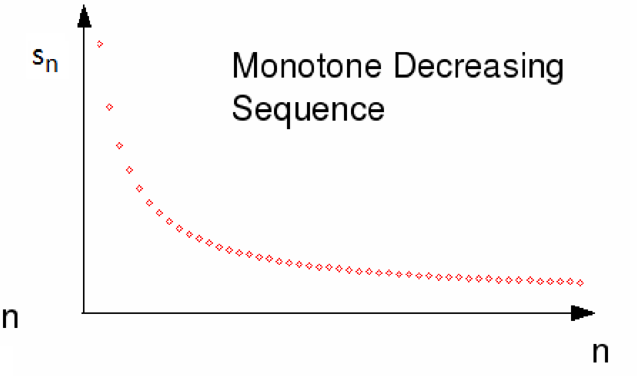
Decreasing Sequence

A sequence $\{s_n\}$ is said to be an decreasing sequence if $s_{n+1} \leq s_n \quad \forall\ n \geq 1$.

Monotonic Sequence

A sequence $\{s_n\}$ is said to be monotonic sequence if it is either increasing or decreasing.





Remarks:

- A sequence $\{s_n\}$ is monotonically increasing if $s_{n+1} - s_n \geq 0$.
- A positive term sequence $\{s_n\}$ is monotonically increasing if $\dfrac{s_{n+1}}{s_n} \geq 1, \;\; \forall \;\; n \geq 1$.
- A sequence $\{s_n\}$ is monotonically decreasing if $s_n - s_{n+1} \geq 0$.
- A positive term sequence $\{s_n\}$ is monotonically decreasing if $\dfrac{s_n}{s_{n+1}} \geq 1 \,, \;\; \forall \;\; n \geq 1$.

Strictly Increasing or Decreasing

A sequence $\{s_n\}$ is called strictly increasing or decreasing according as

$$s_{n+1} > s_n \;\text{ or }\; s_{n+1} < s_n \quad \forall \;\; n \geq 1.$$

Examples:

- $\{n\} = \{1, 2, 3, \ldots\}$ is an increasing sequence.
- $\left\{\dfrac{1}{n}\right\}$ is a decreasing sequence.
- $\{\cos n\pi\} = \{-1, 1, -1, 1, \ldots\}$ is neither increasing nor decreasing.

Questions: 1) Prove that $\left\{1 + \dfrac{1}{n}\right\}$ is a decreasing sequence.

2) Is $\left\{\dfrac{n+1}{n+2}\right\}$ is increasing or decreasing sequence?

Bounded Sequence

A sequence is said to be bounded if its range is a bounded set.

Definition

A sequence $\{s_n\}$ is said to be bounded if there is a number λ so that

$$|s_n| < \lambda \quad \forall \;\; n \in \mathbb{N}.$$

For such a sequence, every term belongs to the interval $[-\lambda, \lambda]$.

It can be noted that if the sequence is bounded then its supremum and infimum exist. If S and s are the supremum and infimum of the bounded sequence $\{s_n\}$, then we write $S = \sup s_n$ and $s = \inf s_n$.

Examples

(i) $\{u_n\} = \left\{\dfrac{(-1)^n}{n}\right\}$ is a bounded sequence

(ii) $\{v_n\} = \{\sin nx\}$ is also bounded sequence. Its supremum is 1 and infimum is -1.

(iii) The geometric sequence $\{ar^{n-1}\}$, $r > 1$ is an unbounded above sequence. It is bounded below by a.

(iv) $\left\{\tan \dfrac{n\pi}{2}\right\}$ is an unbounded sequence.

Convergence of the sequence

The sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$$

is getting closer and closer to the number 0. We say that this sequence converges to 0 or that the limit of the sequence is the number 0. How should this idea be properly defined?

The study of convergent sequences was undertaken and developed in the eighteenth century without any precise definition. The closest one might find to a definition in the early literature would have been something like

A sequence $\{s_n\}$ converges to a number L if the terms of the sequence get closer and closer to L.

However this is too vague and too weak to serve as definition but a rough guide for the intuition, this is misleading in other respects. What about the sequence

0.1, 0.01, 0.02, 0.001, 0.002, 0.0001, 0.0002, 0.00001, 0.00002, ...?

Surely this should converge to 0 but the terms do not get steadily “closer and closer” but back off a bit at each second step.

The definition that captured the idea in the best way was given by Augustin Cauchy in the 1820s. He found a formulation that expressed the idea of “arbitrarily close” using inequalities.

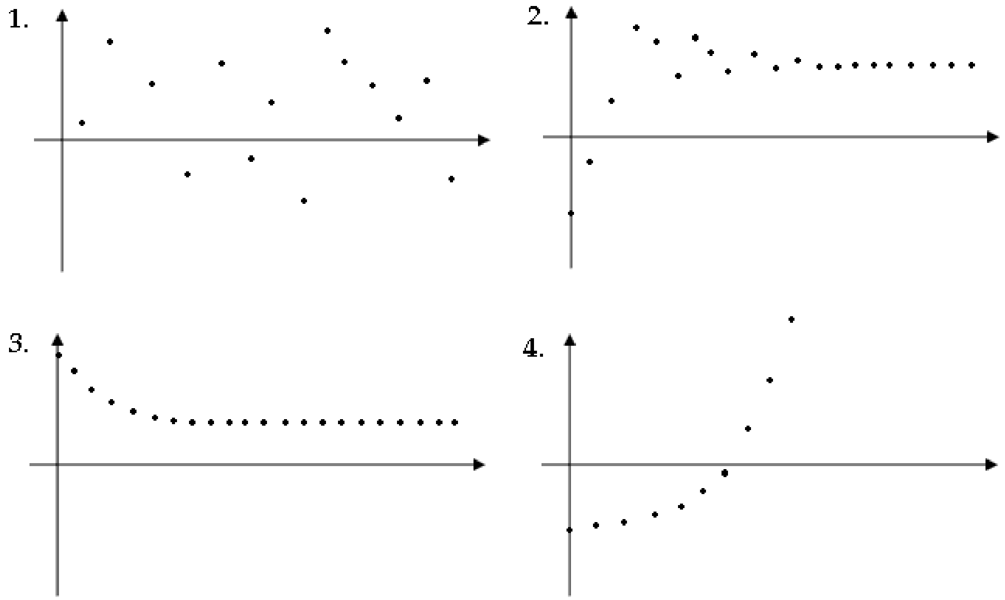
Definition

A sequence $\{s_n\}$ of real numbers is said to convergent to limit ‘s’ as $n \to \infty$, if for every real number $\varepsilon > 0$, there exists a positive integer n_0, depending on ε, so that

$$|s_n - s| < \varepsilon \quad \text{whenever } n > n_0.$$

A sequence that converges is said to be *convergent*. A sequence that fails to converge is said to *divergent*.

We will try to understand it by graph of some sequence. Graph of any four sequences is drawn in the picture below.



Theorem

A convergent sequence of real number has one and only one limit (i.e. limit of the sequence is unique.)

Proof:

Suppose $\{s_n\}$ converges to two limits s and t, where $s \neq t$.

Put $\varepsilon = \dfrac{|s-t|}{2}$ then there exits two positive integers n_1 and n_2 such that

$$|s_n - s| < \varepsilon \qquad \forall \;\; n > n_1$$

and $$|s_n - t| < \varepsilon \qquad \forall \;\; n > n_2.$$

$\Rightarrow$ $|s_n - s| < \varepsilon$ and $|s_n - t| < \varepsilon$ hold simultaneously $\forall \;\; n > \max(n_1, n_2)$.

Thus for all $n > \max(n_1, n_2)$ we have

$$\begin{aligned} |s-t| &= |s - s_n + s_n - t| \\ &\leq |s_n - s| + |s_n - t| \\ &< \varepsilon + \varepsilon = 2\varepsilon, \end{aligned}$$

$$\Rightarrow \; |s-t| < 2\left(\frac{|s-t|}{2}\right),$$

$$\Rightarrow \; |s-t| < |s-t|,$$

which is impossible, therefore the limit of the sequence is unique. ❑

Theorem (Sandwich Theorem or Squeeze Theorem)

Suppose that $\{s_n\}$ and $\{t_n\}$ be two convergent sequences such that $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} t_n$. If $s_n < u_n < t_n \quad \forall\ n \geq n_0$, then the sequence $\{u_n\}$ also converges to *s*.

Proof:

Since the sequence $\{s_n\}$ and $\{t_n\}$ converge to the same limit *s* (say), therefore for given $\varepsilon > 0$ there exists two positive integers $n_1,\ n_2 > n_0$ such that

$$\left| s_n - s \right| < \varepsilon \qquad \forall\ n > n_1,$$
$$\left| t_n - s \right| < \varepsilon \qquad \forall\ n > n_2.$$

i.e.
$$s - \varepsilon < s_n < s + \varepsilon \qquad \forall\ n > n_1,$$
$$s - \varepsilon < t_n < s + \varepsilon \qquad \forall\ n > n_2.$$

Since we have given

$$s_n < u_n < t_n \qquad \forall\ n > n_0$$
$$\therefore\ s - \varepsilon < s_n < u_n < t_n < s + \varepsilon \qquad \forall\ n > \max(n_0, n_1, n_2)$$
$$\Rightarrow\ s - \varepsilon < u_n < s + \varepsilon \qquad \forall\ n > \max(n_0, n_1, n_2)$$

i.e. $\left| u_n - s \right| < \varepsilon \qquad \forall\ n > \max(n_0, n_1, n_2)$

i.e. $\lim\limits_{n\to\infty} u_n = s$. ❑

Example

Show that $\lim\limits_{n\to\infty} \left(\dfrac{1}{(n+1)^2} + \dfrac{1}{(n+2)^2} + \ldots\ldots\ldots\ldots + \dfrac{1}{(2n)^2} \right) = 0$

Solution

Consider

$$s_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots + \frac{1}{(2n)^2}$$

As $\underbrace{\dfrac{1}{(2n)^2} + \dfrac{1}{(2n)^2} + \ldots + \dfrac{1}{(2n)^2}}_{n\ times} \leq s_n < \underbrace{\dfrac{1}{n^2} + \dfrac{1}{n^2} + \ldots + \dfrac{1}{n^2}}_{n\ times}$,

that is,

$$n \cdot \frac{1}{(2n)^2} \ \leq\ s_n \ <\ n \cdot \frac{1}{n^2}$$
$$\Rightarrow\ \frac{1}{4n} \ \leq\ s_n \ <\ \frac{1}{n}$$
$$\Rightarrow\ \lim_{n\to\infty} \frac{1}{4n} \ \leq\ \lim_{n\to\infty} s_n \ <\ \lim_{n\to\infty} \frac{1}{n}$$
$$\Rightarrow\ 0 \ \leq\ \lim_{n\to\infty} s_n \ <\ 0$$
$$\Rightarrow\ \lim_{n\to\infty} s_n = 0$$
❑

Cauchy Sequence

A sequence $\{s_n\}$ of real number is said to be a *Cauchy sequence* if for given number $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$ such that

$$|s_n - s_m| < \varepsilon \qquad \forall\ m,n > n_0$$

Theorem

A Cauchy sequence of real numbers is bounded.

Proof:

Let $\{s_n\}$ be a Cauchy sequence.

Take $\varepsilon = 1$, then there exits a positive integers n_0 such that

$$|s_n - s_m| < 1 \qquad \forall\ m,n > n_0.$$

Fix $m = n_0 + 1$ then

$$\begin{aligned} |s_n| &= |s_n - s_{n_0+1} + s_{n_0+1}| \\ &\le |s_n - s_{n_0+1}| + |s_{n_0+1}| \\ &< 1 + |s_{n_0+1}| \qquad \forall\ n > n_0 \\ &= \lambda \qquad \forall\ n > 1 \text{, and } \lambda = 1 + |s_{n_0+1}| \quad (n_0 \text{ changes as } \varepsilon \text{ changes}) \end{aligned}$$

Hence we conclude that $\{s_n\}$ is a Cauchy sequence, which is bounded one. ❑

Note:

(i) Convergent sequence is bounded.

(ii) The converse of the above theorem does not hold.

i.e. every bounded sequence is not Cauchy.

Consider the sequence $\{s_n\}$ where $s_n = (-1)^n$, $n \ge 1$. It is bounded sequence because

$$|(-1)^n| = 1 < 2 \qquad \forall\ n \ge 1.$$

But it is not a Cauchy sequence if it is then for $\varepsilon = 1$ we should be able to find a positive integer n_0 such that $|s_n - s_m| < 1$ for all $m,n > n_0$.

But with $m = 2k+1$, $n = 2k+2$ when $2k+1 > n_0$, we arrive at

$$\begin{aligned} |s_n - s_m| &= |(-1)^{2n+2} - (-1)^{2k+1}| \\ &= |1+1| = 2 < 1 \quad \text{is absurd.} \end{aligned}$$

Hence $\{s_n\}$ is not a Cauchy sequence. Also this sequence is not a convergent sequence. (it is an oscillatory sequence).

Question: Prove that every Cauchy sequence of real number is bounded but converse is not true.

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Theorem

If the sequence $\{s_n\}$ converges to s then $\exists$ a positive integer n such that $|s_n| > \frac{1}{2}s$.

Proof:

We fix $\varepsilon = \frac{1}{2}|s| > 0$

$\Rightarrow$ $\exists$ a positive integer n_1 such that

$$|s_n - s| < \varepsilon \qquad \text{for } n > n_1$$

$$\Rightarrow |s_n - s| < \frac{1}{2}|s|$$

Now

$$\frac{1}{2}|s| = |s| - \frac{1}{2}|s|$$

$$< |s| - |s_n - s| \;\; \le |s + (s_n - s)|$$

$$\Rightarrow \frac{1}{2}|s| < |s_n|.$$ ❑

Theorem

Let a and b be fixed real numbers if $\{s_n\}$ and $\{t_n\}$ converge to s and t respectively, then

(i) $\{as_n + bt_n\}$ converges to $as + bt$.

(ii) $\{s_n t_n\}$ converges to st.

(iii) $\left\{\frac{s_n}{t_n}\right\}$ converges to $\frac{s}{t}$, provided $t_n \ne 0 \;\; \forall \; n$ and $t \ne 0$.

Proof:

Since $\{s_n\}$ and $\{t_n\}$ converge to s and t respectively,

$$\therefore \; |s_n - s| < \varepsilon \qquad \forall \; n > n_1 \in \mathbb{N}$$

$$|t_n - t| < \varepsilon \qquad \forall \; n > n_2 \in \mathbb{N}$$

Also $\exists$ $\lambda > 0$ such that $|s_n| < \lambda \quad \forall \; n > 1$ $\qquad$ ($\because$ $\{s_n\}$ is bounded)

(i) We have

$$\begin{aligned} |(as_n + bt_n) - (as + bt)| &= |a(s_n - s) + b(t_n - t)| \\ &\le |a(s_n - s)| + |b(t_n - t)| \\ &< |a|\varepsilon + |b|\varepsilon \qquad \forall \; n > \max(n_1, n_2) \\ &= \varepsilon_1, \end{aligned}$$

where $\varepsilon_1 = |a|\varepsilon + |b|\varepsilon$ a certain number.

This implies $\{as_n + bt_n\}$ converges to $as + bt$.

(*ii*) $\left|s_n t_n - st\right| = \left|s_n t_n - s_n t + s_n t - st\right|$

$$= \left|s_n\left(t_n - t\right) + t\left(s_n - s\right)\right| \quad \le \left|s_n\right| \cdot \left|\left(t_n - t\right)\right| + \left|t\right| \cdot \left|\left(s_n - s\right)\right|$$

$$< \lambda\varepsilon + \left|t\right|\varepsilon \qquad \forall \ n > \max(n_1, n_2)$$

$$= \varepsilon_2, \qquad \text{where } \varepsilon_2 = \lambda\varepsilon + \left|t\right|\varepsilon \text{ a certain number.}$$

This implies $\{s_n t_n\}$ converges to *st*.

(*iii*) $\left|\dfrac{1}{t_n} - \dfrac{1}{t}\right| = \left|\dfrac{t - t_n}{t_n t}\right|$

$$= \frac{\left|t_n - t\right|}{\left|t_n\right|\left|t\right|} < \frac{\varepsilon}{\frac{1}{2}\left|t\right|\left|t\right|} \qquad \forall \ n > \max(n_1, n_2) \qquad \because \ \left|t_n\right| > \frac{1}{2}t$$

$$= \frac{\varepsilon}{\frac{1}{2}\left|t\right|^2} = \varepsilon_3, \qquad \text{where } \varepsilon_3 = \frac{\varepsilon}{\frac{1}{2}\left|t\right|^2} \text{ a certain number.}$$

This implies $\left\{\dfrac{1}{t_n}\right\}$ converges to $\dfrac{1}{t}$.

Hence $\left\{\dfrac{s_n}{t_n}\right\} = \left\{s_n \cdot \dfrac{1}{t_n}\right\}$ converges to $s \cdot \dfrac{1}{t} = \dfrac{s}{t}$. (from (ii)) ❑

Theorem

For each irrational number *x*, there exists a sequence $\{r_n\}$ of distinct rational numbers such that $\lim\limits_{n\to\infty} r_n = x$.

Proof:

Since *x* and *x* + 1 are two different real numbers

$\because$ $\exists$ a rational number r_1 such that

$$x < r_1 < x + 1$$

Similarly $\exists$ a rational number $r_2 \ne r_1$ such that

$$x < r_2 < \min\left(r_1, x + \frac{1}{2}\right) < x + 1$$

Continuing in this manner we have

$$x < r_3 < \min\left(r_2, x + \frac{1}{3}\right) < x + 1$$

$$x < r_4 < \min\left(r_3, x + \frac{1}{4}\right) < x + 1$$

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$$x < r_n < \min\left(r_{n-1}, x + \frac{1}{n}\right) < x + 1$$

This implies that $\exists$ a sequence $\{r_n\}$ of the distinct rational number such that

$$x < r_n < x + \frac{1}{n}.$$

Since

$$\lim_{n\to\infty}(x) = \lim_{n\to\infty}\left(x + \frac{1}{n}\right) = x.$$

Therefore

$$\lim_{n\to\infty} r_n = x.$$ ❑

Theorem

Let a sequence $\{s_n\}$ be a bounded sequence.

(i) If $\{s_n\}$ is monotonically increasing then it converges to its supremum.

(ii) If $\{s_n\}$ is monotonically decreasing then it converges to its infimum.

Proof

(i) Let $S = \sup s_n$ and take $\varepsilon > 0$.

$\therefore\ \exists\ s_{n_0}$ such that $S - \varepsilon < s_{n_0}$

Since $\{s_n\}$ is monotonically increasing,

$\therefore\ S - \varepsilon < s_{n_0} < s_n < S < S + \varepsilon$ for $n > n_0$

$\Rightarrow\ S - \varepsilon < s_n < S + \varepsilon$ for $n > n_0$

$\Rightarrow\ |s_n - S| < \varepsilon$ for $n > n_0$

$\Rightarrow\ \lim\limits_{n\to\infty} s_n = S$

(ii) Let $s = \inf s_n$ and take $\varepsilon > 0$.

$\therefore\ \exists\ s_{n_1}$ such that $s_{n_1} < s + \varepsilon$

Since $\{s_n\}$ is monotonically decreasing,

$\therefore\ s - \varepsilon < s < s_n < s_{n_1} < s + \varepsilon$ for $n > n_1$

$\Rightarrow\ s - \varepsilon < s_n < s + \varepsilon$ for $n > n_1$

$\Rightarrow\ |s_n - s| < \varepsilon$ for $n > n_1$

Thus $\lim\limits_{n\to\infty} s_n = s$ ❑

Note

A monotonic sequence can not oscillate infinitely.

Question:

Let $\{s_n\}$ be a sequence and $\lim\limits_{n\to\infty} s_n = s$. Then prove that $\lim\limits_{n\to\infty} s_{n+1} = s$.

Recurrence Relation

A sequence is said to be defined *recursively* or *by recurrence relation* if the general term is given as a relation of its preceding and succeeding terms in the sequence together with some initial condition.

Example:

Let $t_1 > 1$ and let $\{t_n\}$ be defined by $t_{n+1} = 2 - \dfrac{1}{t_n}$ for $n \geq 1$.

(i) Show that $\{t_n\}$ is decreasing sequence.

(ii) It is bounded below.

(iii) Find the limit of the sequence.

Since $t_1 > 1$ and $\{t_n\}$ is defined by $t_{n+1} = 2 - \dfrac{1}{t_n}$; $n \geq 1$

$$\Rightarrow t_n > 0 \quad \forall \; n \geq 1$$

Also $t_n - t_{n+1} = t_n - 2 + \dfrac{1}{t_n}$

$$= \frac{t_n^2 - 2t_n + 1}{t_n} = \frac{(t_n - 1)^2}{t_n} > 0.$$

$$\Rightarrow t_n > t_{n+1} \quad \forall \; n \geq 1.$$

This implies that t_n is monotonically decreasing.

Since $t_n > 1 \quad \forall \; n \geq 1,$

$\Rightarrow t_n$ is bounded below.

Since t_n is decreasing and bounded below therefore t_n is convergent.

Let us suppose $\lim\limits_{n\to\infty} t_n = t$.

Then $\lim\limits_{n\to\infty} t_{n+1} = \lim\limits_{n\to\infty} t_n \quad \Rightarrow \lim\limits_{n\to\infty}\left(2 - \dfrac{1}{t_n}\right) = \lim\limits_{n\to\infty} t_n$

$$\Rightarrow 2 - \frac{1}{t} = t \quad \Rightarrow \frac{2t-1}{t} = t \quad \Rightarrow 2t - 1 = t^2 \quad \Rightarrow t^2 - 2t + 1 = 0$$

$$\Rightarrow (t-1)^2 = 0 \quad \Rightarrow t = 1.$$

❑

Question:

- Let $\{t_n\}$ be a positive term sequence. Find the limit of the sequence if $4t_{n+1} = \dfrac{2}{5} - 3t_n$ for all $n \geq 1$.
- Let $\{u_n\}$ be a sequence of positive numbers. Then find the limit of the sequence if $u_{n+1} = \dfrac{1}{u_n} + \dfrac{1}{4}u_{n-1}$ for $n \geq 1$.
- The Fibonacci numbers are: $F_1 = F_2 = 1$, and for every $n \geq 3$, F_n is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$. Find the $\lim\limits_{n\to\infty} \dfrac{F_n}{F_{n-1}}$ (this limit is known as golden number)

Theorem

Every Cauchy sequence of real numbers has a convergent subsequence.

Proof:

Suppose $\{s_n\}$ is a Cauchy sequence.

Let $\varepsilon > 0$ then $\exists$ a positive integer $n_0 \geq 1$ such that

$$\left|s_{n_k} - s_{n_{k-1}}\right| < \frac{\varepsilon}{2^k} \qquad \forall \; n_k, n_{k-1}, \; k = 1,2,3,.........$$

Put $\quad b_k = \left(s_{n_1} - s_{n_0}\right) + \left(s_{n_2} - s_{n_1}\right) + \ldots + \left(s_{n_k} - s_{n_{k-1}}\right)$

$$\Rightarrow \left|b_k\right| = \left|\left(s_{n_1} - s_{n_0}\right) + \left(s_{n_2} - s_{n_1}\right) + \ldots + \left(s_{n_k} - s_{n_{k-1}}\right)\right|$$

$$\leq \left|\left(s_{n_1} - s_{n_0}\right)\right| + \left|\left(s_{n_2} - s_{n_1}\right)\right| + \ldots + \left|\left(s_{n_k} - s_{n_{k-1}}\right)\right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \ldots + \frac{\varepsilon}{2^k}$$

$$= \varepsilon\left(\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^k}\right) \quad = \varepsilon\left(\frac{\frac{1}{2}\left(1 - \frac{1}{2^k}\right)}{1 - \frac{1}{2}}\right) = \varepsilon\left(1 - \frac{1}{2^k}\right)$$

$$\Rightarrow \left|b_k\right| < \varepsilon \qquad \forall \; k \geq 1$$

$\Rightarrow \quad \{b_k\}$ is convergent

$$\because \;\; b_k = s_{n_k} - s_{n_0} \qquad \therefore \;\; s_{n_k} = b_k + s_{n_0},$$

where s_{n_0} is a certain fix number therefore $\left\{s_{n_k}\right\}$ which is a subsequence of $\{s_n\}$ is convergent. ❑

Theorem (Cauchy's General Principle for Convergence)

A sequence of real number is convergent if and only if it is a Cauchy sequence.

Proof:

<u>Necessary Condition</u>

Let $\{s_n\}$ be a convergent sequence, which converges to s.

Then for given $\varepsilon > 0$ $\exists$ a positive integer n_0, such that

$$\left|s_n - s\right| < \frac{\varepsilon}{2} \quad \forall \;\; n > n_0$$

Now for $n > m > n_0$

$$\left|s_n - s_m\right| = \left|s_n - s + s - s_m\right|$$

$$\leq \left|s_n - s\right| + \left|s - s_m\right| = \left|s_n - s\right| + \left|s_m - s\right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Which shows that $\{s_n\}$ is a Cauchy sequence.

Sufficient Condition

Let us suppose that $\{s_n\}$ is a Cauchy sequence then for $\varepsilon > 0$, $\exists$ a positive integer m_1 such that

$$|s_n - s_m| < \frac{\varepsilon}{2} \;\; \forall \;\; n,m > m_1 \;\; \ldots\ldots\ldots\ldots (i)$$

Since $\{s_n\}$ is a Cauchy sequence

therefore it has a subsequence $\{s_{n_k}\}$ converging to s (say).

$\Rightarrow$ $\exists$ a positive integer m_2 such that

$$\left|s_{n_k} - s\right| < \frac{\varepsilon}{2} \qquad \forall \;\; n > m_2 \;\; \ldots\ldots\ldots\ldots (ii)$$

Now

$$\begin{aligned} |s_n - s| &= \left|s_n - s_{n_k} + s_{n_k} - s\right| \\ &\le \left|s_n - s_{n_k}\right| + \left|s_{n_k} - s\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \forall \;\; n > \max(m_1, m_2), \end{aligned}$$

this shows that $\{s_n\}$ is a convergent sequence. ❑

Example

Prove that $\left\{1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots\ldots\ldots + \frac{1}{n}\right\}$ is divergent sequence.

Let $\{t_n\}$ be defined by

$$t_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots\ldots\ldots + \frac{1}{n}.$$

For $m, n \in \mathbb{N}$, $n > m$ we have

$$\begin{aligned} |t_n - t_m| &= \frac{1}{m+1} + \frac{1}{m+2} + \ldots\ldots\ldots\ldots + \frac{1}{n} \\ &> (n-m)\frac{1}{n} \;\; = \;\; 1 - \frac{m}{n}. \end{aligned}$$

In particular if $n = 2m$ then

$$|t_n - t_m| > \frac{1}{2}.$$

This implies that $\{t_n\}$ is not a Cauchy sequence therefore it is divergent. ❑

Theorem (nested intervals)

Suppose that $\{I_n\}$ is a sequence of the closed interval such that $I_n = [a_n, b_n]$, $I_{n+1} \subset I_n \;\; \forall \;\; n \ge 1$, and $(b_n - a_n) \to 0$ as $n \to \infty$ then $\bigcap I_n$ contains one and only one point.

Proof:

Since $I_{n+1} \subset I_n$

$\therefore\ a_1 < a_2 < a_3 < \ldots < a_{n-1} < a_n < b_n < b_{n-1} < \ldots < b_3 < b_2 < b_1$

$\{a_n\}$ is increasing sequence, bounded above by b_1 and bounded below by a_1.

And $\{b_n\}$ is decreasing sequence bounded below by a_1 and bounded above by b_1.

$\Rightarrow$ $\{a_n\}$ and $\{b_n\}$ both are convergent.

Suppose $\{a_n\}$ converges to a and $\{b_n\}$ converges to b.

But $|a-b| = |a - a_n + a_n - b_n + b_n - b|$

$\leq |a_n - a| + |a_n - b_n| + |b_n - b| \to 0$ as $n \to \infty$.

$\Rightarrow a = b$

and $a_n < a < b_n \quad \forall\ n \geq 1$. ❑

Limit Inferior of the sequence

Suppose $\{s_n\}$ is bounded below then we define limit inferior of $\{s_n\}$ as follow

$$\lim_{n\to\infty}(\inf s_n) = \lim_{n\to\infty} u_k,\quad \text{where}\quad u_k = \inf\{s_n : n \geq k\}$$

If s_n is not bounded below then

$$\lim_{n\to\infty}(\inf s_n) = -\infty.$$

Limit Superior of the sequence

Suppose $\{s_n\}$ is bounded above then we define limit superior of $\{s_n\}$ as follow

$$\lim_{n\to\infty}(\sup s_n) = \lim_{n\to\infty} v_k,\quad \text{where}\quad v_k = \sup\{s_n : n \geq k\}$$

If s_n is not bounded above then we have

$$\lim_{n\to\infty}(\sup s_n) = +\infty.$$

Note:

(i) A bounded sequence has unique limit inferior and superior

(ii) Let $\{s_n\}$ contains all the rational numbers, then every real number is a subsequencial limit then limit superior of s_n is $+\infty$ and limit inferior of s_n is $-\infty$

(iii) Let $\{s_n\} = (-1)^n\left(1 + \dfrac{1}{n}\right)$

then limit superior of s_n is 1 and limit inferior of s_n is -1.

(iv) Let $s_n = \left(1 + \dfrac{1}{n}\right)\cos n\pi$.

Then $u_k = \inf\{s_n : n \geq k\}$

$$= \inf\left\{\left(1 + \frac{1}{k}\right)\cos k\pi, \left(1 + \frac{1}{k+1}\right)\cos(k+1)\pi, \left(1 + \frac{1}{k+2}\right)\cos(k+2)\pi, \ldots\right\}$$

$$= \begin{cases} \left(1+\dfrac{1}{k}\right)\cos k\pi & \textit{if } k \textit{ is odd} \\ \left(1+\dfrac{1}{k+1}\right)\cos (k+1)\pi & \textit{if } k \textit{ is even} \end{cases}$$

$$\Rightarrow \lim_{n\to\infty}\left(\inf s_n\right) = \lim_{n\to\infty} u_k = -1$$

Also $\quad v_k = \sup\{s_n : n \geq k\}$

$$= \begin{cases} \left(1+\dfrac{1}{k+1}\right)\cos (k+1)\pi & \textit{if } k \textit{ is odd} \\ \left(1+\dfrac{1}{k}\right)\cos k\pi & \textit{if } k \textit{ is even} \end{cases}$$

$$\Rightarrow \lim_{n\to\infty}\left(\sup s_n\right) = \lim_{n\to\infty} v_k = 1$$ ❑

Theorem

If $\{s_n\}$ is a convergent sequence then

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\inf s_n\right) = \lim_{n\to\infty}\left(\sup s_n\right)$$

Proof:

Let $\lim\limits_{n\to\infty} s_n = s$ then for a real number $\varepsilon > 0$, $\exists$ a positive integer n_0 such that

$$\left|s_n - s\right| < \varepsilon \qquad \forall\ n \geq n_0 \ \ldots\ldots\ldots\ (i)$$

i.e. $\quad s-\varepsilon < s_n < s+\varepsilon \qquad \forall\ n \geq n_0$

If $\quad v_k = \sup\{s_n : n \geq k\}$

Then $\quad s-\varepsilon < v_n < s+\varepsilon \qquad \forall\ k \geq n_0$

$$\Rightarrow s-\varepsilon < \lim_{k\to\infty} v_n < s+\varepsilon \qquad \forall\ k \geq n_0 \ \ldots\ldots\ldots\ldots\ (ii)$$

from (*i*) and (*ii*) we have

$$s = \lim_{k\to\infty}\ \sup\{s_n\}$$

We can have the same result for limit inferior of $\{s_n\}$ by taking

$$u_k = \inf\{s_n : n \geq k\}\ .$$ ❑

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