Chapter 5 – Differentiation

Course Title: Real Analysis 1 Course Code: MTH321
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❖ *Derivative of a function:*

Let f be defined and real valued on (a,b). For any point $c \in (a,b)$, form the quotient

$$\frac{f(x)-f(c)}{x-c}.$$

We fix point c and study the behaviour of this quotient as $x \to c$.

❖ *Definition*

Let f be defined on an open interval (a,b), and assume that $c \in (a,b)$. Then f is said to be differentiable at c whenever the limit

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

exists. This limit is denoted by $f'(c)$ and is called the derivative of f at point c.

If $x-c=h$, then we have

$$f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}.$$

❖ *Example*

(i) A function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases} x^2\sin\frac{1}{x} & ; \ x\neq 0 \\ 0 & ; \ x=0 \end{cases}$$

This function is differentiable at $x=0$ because

$$\begin{aligned}\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0}\frac{x^2\sin\frac{1}{x}-0}{x-0} \\ &= \lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{x} = \lim_{x\to 0} x\sin\frac{1}{x} = 0\end{aligned}$$

(ii) Let $f(x)=x^n$; $n\geq 0$ (n is integer), $x\in\mathbb{R}$.
Then

$$\begin{aligned}\lim_{x\to c}\frac{f(x)-f(c)}{x-c} &= \lim_{x\to c}\frac{x^n-c^n}{x-c} \\ &= \lim_{x\to c}\frac{(x-c)(x^{n-1}+cx^{n-2}+\ldots+c^{n-2}x+c^{n-1})}{x-c} \\ &= \lim_{x\to c}(x^{n-1}+cx^{n-2}+\ldots+c^{n-2}x+c^{n-1}) \\ &= nc^{n-1}\end{aligned}$$

implies that f is differentiable every where and $f'(x)=nx^{n-1}$.

❖ ***Theorem***

Let f be defined on (a,b), if f is differentiable at a point $x\in(a,b)$, then f is continuous at *x*. (Differentiability implies continuity)

Proof

We know that

$$\lim_{t\to x}\frac{f(t)-f(x)}{t-x}=f'(x) \qquad \text{where } t\neq x \quad \text{and} \quad a<t<b$$

Now

$$\lim_{t\to x}\big(f(t)-f(x)\big)=\lim_{t\to x}\left(\frac{f(t)-f(x)}{t-x}\right)\lim_{t\to x}(t-x)$$
$$=f'(x)\cdot 0$$
$$=0$$
$$\Rightarrow \lim_{t\to x}f(t)=f(x).$$

This show that *f* is continuous at *x*.

❖ ***Remarks***

(*i*) The converse of the above theorem does not hold.

Consider $f(x)=|x|=\begin{cases} x & if \ \ x\geq 0 \\ -x & if \ \ x<0 \end{cases}$

$f'(0)$ does not exists but $f(x)$ is continuous at $x=0$

(*ii*) If *f* is discontinuous at $c\in\mathbf{D}_f$ then $f'(c)$ does not exist.

e.g.

$$f(x)=\begin{cases} 1 & if \ \ x>0 \\ 0 & if \ \ x\leq 0 \end{cases}$$

is discontinuous at $x=0$ therefore it is not differentiable at $x=0$.

❖ ***Theorem***

Suppose f and g are defined on $[a,b]$ and are differentiable at a point $x\in[a,b]$, then $f+g$, fg and $\frac{f}{g}$ are differentiable at *x* and

(*i*) $(f+g)'(x)=f'(x)+g'(x)$

(*ii*) $(fg)'(x)=f'(x)g(x)+f(x)g'(x)$

(*iii*) $\left(\frac{f}{g}\right)'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}$

The proof of this theorem can be get from any F.Sc or B.Sc text book.

❖ ***Theorem (Chain Rule)***

Suppose f is continuous on $[a,b]$, $f'(x)$ exists at some point $x \in [a,b]$. A function g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$.

If $\quad h(t) = g\big(f(t)\big)$; $a \le t \le b$

Then h is differentiable at x and $h'(x) = g'\big(f(x)\big)\cdot f'(x)$.

Proof

Let $y = f(x)$

By the definition of the derivative we have

$$f(t) - f(x) = (t-x)\big[f'(x) + u(t)\big] \quad \ldots\ldots\ldots\ldots (i)$$

and $$g(s) - g(y) = (s-y)\big[g'(y) + v(s)\big] \quad \ldots\ldots\ldots\ldots (ii)$$

where $t \in [a,b]$, $s \in I$ and $u(t) \to 0$ as $t \to x$ and $v(s) \to 0$ as $s \to y$.

Let us suppose $s = f(t)$ then

$$\begin{aligned} h(t) - h(x) &= g\big(f(t)\big) - g\big(f(x)\big) \\ &= \big[f(t) - f(x)\big]\big[g'(y) + v(s)\big] && \text{by } (ii) \\ &= (t-x)\big[f'(x) + u(t)\big]\big[g'(y) + v(s)\big] && \text{by } (i) \end{aligned}$$

or if $t \ne x$

$$\frac{h(t) - h(x)}{t - x} = \big[f'(x) + u(t)\big]\big[g'(y) + v(s)\big]$$

taking the limit as $t \to x$ we have

$$\begin{aligned} h'(x) &= \big[f'(x) + 0\big]\big[g'(y) + 0\big] \\ &= g'\big(f(x)\big)\cdot f'(x) && \because \; y = f(x) \end{aligned}$$

which is the required result.

It is known as *chain rule*.

❖ ***Example***

Let us find the derivative of $\sin(2x)$, One way to do that is through some trigonometric identities. Indeed, we have

$$\sin(2x) = 2\sin(x)\cos(x)\cdot$$

So we will use the product formula to get

$$\big(\sin(2x)\big)' = 2\big(\sin'(x)\cos(x) + \sin(x)\cos'(x)\big)$$

which implies

$$\big(\sin(2x)\big)' = 2\big(\cos^2(x) - \sin^2(x)\big)\cdot$$

Using the trigonometric formula $\cos(2x) = \cos^2(x) - \sin^2(x)$, we have

$$\big(\sin(2x)\big)' = 2\cos(2x)\cdot$$

Once this is done, you may ask about the derivative of $\sin(5x)$? The answer can be found using similar trigonometric identities, but the calculations are not as easy as before. We will see how the Chain Rule formula will answer this question in an elegant way.

Let us find the derivative of $\sin(5x)$.

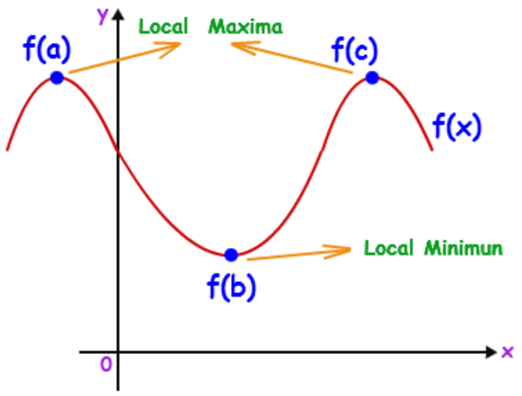
We have $h(x) = f(g(x))$, where $g(x) = 5x$ and $f(x) = \sin(x)$. Then the Chain rule implies that $h'(x)$ exists and

$$h'(x) = 5 \cdot [\cos(5x)] = 5\cos(5x).$$

❖ *Local Maximum*

Let f be a real valued function defined on a metric space X, we say that f has a local maximum at a point $p \in X$ if there exist $\delta > 0$ such that $f(x) \le f(p)\ \forall\ x \in X$ with $|x - p| < \delta$.

Local minimum is defined likewise.



❖ *Theorem*

Let f be defined on $[a,b]$, if f has a local maximum at a point $x \in [a,b]$ and if $f'(x)$ exist then $f'(x) = 0$.

(The analogous for local minimum is of course also true)

Proof

Choose δ such that

$$a < x - \delta < x < x + \delta < b$$

Now if $x - \delta < t < x$ then

$$\frac{f(t) - f(x)}{t - x} \ge 0$$

Taking limit as $t \to x$ we get

$$f'(x) \ge 0 \ \ldots\ldots\ldots\ldots\ (i)$$

If $x < t < x + \delta$

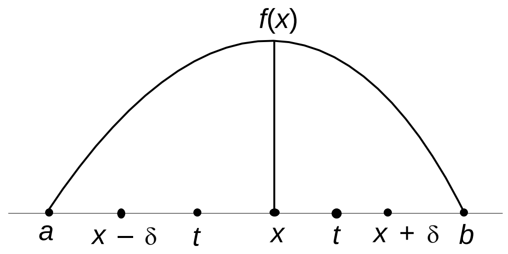
Then

$$\frac{f(t) - f(x)}{t - x} \le 0$$

Again taking limit when $t \to x$ we get

$$f'(x) \le 0 \ \ldots\ldots\ldots\ldots\ldots\ (ii)$$

Combining (i) and (ii) we have $f'(x) = 0$.



❖ *Lagrange's Mean Value Theorem.*

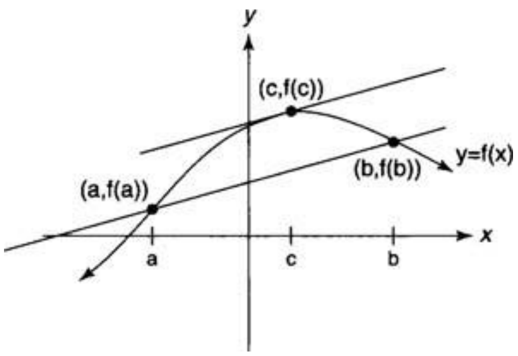
Let f be

i) continuous on $[a,b]$

ii) differentiable on (a,b).

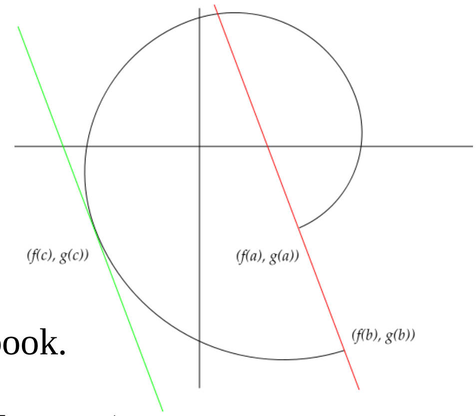
Then there exists a point $c \in (a,b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$



We are skipping the proof as it is included in BSc calculus book.

❖ *Generalized Mean Value Theorem*



If f and g are continuous real valued functions on closed interval $[a,b]$ and f is differentiable on (a,b), then there is a point $c \in (a,b)$ at which

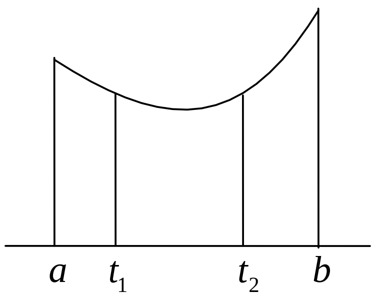
$$[f(b)-f(a)]g'(c)=[g(b)-g(a)]f'(c)$$

The differentiability is not required at the end point.
We are skipping the proof as it is included in BSc calculus book.

❖ *Theorem (Intermediate Value Theorem or Darboux,s Theorem)*

Suppose f is a real differentiable function on some interval (a,b) and suppose $f'(a)<\lambda<f'(b)$ then there exist a point $x\in(a,b)$ such that $f'(x)=\lambda$.
A similar result holds if $f'(a)>f'(b)$.

Proof



Put $g(t)=f(t)-\lambda t$
Then $g'(t)=f'(t)-\lambda$
If $t=a$ we have

$$g'(a)=f'(a)-\lambda$$

$$\because\ f'(a)-\lambda<0 \quad \therefore\ g'(a)<0$$

implies that g is monotonically decreasing at a.
$\Rightarrow\ \exists$ a point $t_1\in(a,b)$ such that $g(a)>g(t_1)$.
Similarly,

$$g'(b)=f'(b)-\lambda$$

$$\because\ f'(b)-\lambda>0 \quad \therefore\ g'(b)>0$$

implies that g is monotonically increasing at b.
$\Rightarrow\ \exists$ a point $t_2\in(a,b)$ such that $g(t_2)<g(b)$
$\Rightarrow$ the function attain its minimum on (a,b) at a point x (say)
such that $g'(x)=0 \ \Rightarrow\ f'(x)-\lambda=0$

$$\Rightarrow\ f'(x)=\lambda.$$

❖ *Question*

Let f be defined for all real x and suppose that
$|f(x)-f(y)|\le(x-y)^2 \quad \forall$ real x & y. Prove that f is constant.

Solution

Since $|f(x)-f(y)|\le(x-y)^2$
Therefore

$$-(x-y)^2\le f(x)-f(y)\le(x-y)^2$$

Dividing throughout by $x-y$, we get

$$-(x-y)\le \frac{f(x)-f(y)}{x-y}\le (x-y) \quad \text{when} \quad x>y$$

and

$$-(x-y)\ge \frac{f(x)-f(y)}{x-y}\ge (x-y) \quad \text{when} \quad x<y$$

Taking limit as $x\to y$, we get

$$\left.\begin{array}{l} 0\le f'(y)\le 0 \\ 0\ge f'(y)\ge 0 \end{array}\right] \quad \Rightarrow \; f'(y)=0$$

which shows that function is constant.

❖ ***Question***

Suppose f is defined and differentiable for every $x>0$ and $f'(x)\to 0$ as $x\to +\infty$ put $g(x)=f(x+1)-f(x)$. Prove that $g(x)\to 0$ as $x\to +\infty$.

Solution

Since f is defined and differentiable for $x>0$ therefore we can apply the Lagrange's M.V. T. to have

$$f(x+1)-f(x)=(x+1-x)f'(x_1) \quad \text{where} \quad x<x_1.$$

$\because \; f'(x)\to 0$ as $x\to\infty$

$\therefore \; f'(x_1)\to 0$ as $x\to\infty$

$\Rightarrow \; f(x+1)-f(x)\to 0$ as $x\to 0$

$\Rightarrow \; g(x)\to 0$ as $x\to 0$.

❖ ***Question (L Hospital Rule)***

Suppose $f'(x),\, g'(x)$ exist, $g'(x)\ne 0$ and $f(x)=g(x)=0$.

Prove that $\lim\limits_{t\to x}\dfrac{f(t)}{g(t)}=\dfrac{f'(x)}{g'(x)}$

Proof

$$\lim_{t\to x}\frac{f(t)}{g(t)}=\lim_{t\to x}\frac{f(t)-0}{g(t)-0}=\lim_{t\to x}\frac{f(t)-f(x)}{g(t)-(x)} \qquad \because \; f(x)=g(x)=0$$

$$=\lim_{t\to x}\frac{f(t)-f(x)}{t-x}\cdot\frac{t-x}{g(t)-(x)}$$

$$=\lim_{t\to x}\frac{f(t)-f(x)}{t-x}\cdot\lim_{t\to x}\frac{1}{\dfrac{g(t)-(x)}{t-x}}$$

$$=\lim_{t\to x}\frac{f(t)-f(x)}{t-x}\cdot\frac{1}{\lim\limits_{t\to x}\dfrac{g(t)-(x)}{t-x}}=f'(x)\cdot\frac{1}{g'(x)}=\frac{f'(x)}{g'(x)}$$

❖ ***Question***

Suppose f is defined in the neighborhood of a point x and $f''(x)$ exists.

Show that $\lim\limits_{h\to 0}\dfrac{f(x+h)+f(x-h)-2f(x)}{h^2}=f''(x)$.

Solution

By use of Lagrange's Mean Value Theorem

$$f(x+h)-f(x)=hf'(x_1) \quad \text{where} \quad x<x_1<x+h \;\ldots\ldots\ldots\ldots\ldots\; (i)$$

and

$$-[f(x-h)-f(x)]=hf'(x_2) \quad \text{where} \quad x-h<x_2<x \;\ldots\ldots\ldots\ldots\ldots\; (ii)$$

Subtract (*ii*) from (*i*) to get

$$f(x+h)+f(x-h)-2f(x)=h[f'(x_1)-f'(x_2)]$$

$$\Rightarrow \frac{f(x+h)+f(x-h)-2f(x)}{h^2}=\frac{f'(x_1)-f'(x_2)}{h}.$$

Since $x_2-x_1\to 0$ as $h\to 0$,

therefore

$$\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}=\lim_{x_1\to x_2}\frac{f'(x_1)-f'(x_2)}{x_1-x_2}$$

$$=f''(x_2)$$

❖ *Question*

If $c_0+\dfrac{c_1}{2}+\dfrac{c_2}{3}+\ldots+\dfrac{c_{n-1}}{n}+\dfrac{c_n}{n+1}=0$

Where $c_0,c_1,c_2,\ldots,c_n$ are real constants.

Prove that $c_0+c_1x+c_2x^2+\ldots+c_nx^n=0$ has at least one real root between 0 and 1.

Solution

Suppose $f(x)=c_0x+\dfrac{c_1}{2}x^2+\ldots+\dfrac{c_n}{n+1}x^{n+1}$

Then $f(0)=0$ and $f(1)=c_0+\dfrac{c_1}{2}+\dfrac{c_2}{3}+\ldots+\dfrac{c_n}{n+1}=0$

$\Rightarrow f(0)=f(1)=0$

$\because f(x)$ is a polynomial therefore we have

i) It is continuous on $[0,1]$

ii) It is differentiable on $(0,1)$

iii) And $f(a)=0=f(b)$

$\Rightarrow$ the function f has local maximum or a local minimum at some point $x\in(0,1)$

$\Rightarrow f'(x)=c_0+c_1x+c_2x^2+\ldots+c_nx^n=0$ for some $x\in(0,1)$

$\Rightarrow$ the given equation has real root between 0 and 1.

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