Basic Concepts of REAL ANALYSIS Part I

Notes from Praveen Chhikara's Lectures

Salient Features:

Tests Included with discussion

Boost up score for JAM & CSIR-NET Exams

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Notes:

1. **CSIR-NET Maths Students**: The **Part 1** of these notes does not contain the full syllabus. It contains some of the important topics, which will definitely help you score well. The other topics are covered in **Part 2** of Real Analysis Notes.

2. **JAM Maths Students**: It contains all topics, but do not rely completely on these notes. Have some standard book to follow.

Important Note: These notes may not contain everything that you are interested in studying. These notes can make your work easier at first. But you should study books. **Nothing can replace books.**

Suggestion: Follow the book "Understanding Analysis by Stephen Abbott " to get much of the notes.

THANKFUL Note: The notes were written beautifully by **Archana Arya**, during my classes, to whom I am very much thankful.

Your suggestions are always welcome for anything; something to be added, some mistakes in the notes, or anything.

Contents

SYLLABI

JAM Mathematics

Real Analysis: Sequence of real numbers, convergence of sequences, bounded and monotone sequences, convergence criteria for sequences of real numbers, Cauchy sequences, subsequences, Bolzano-Weierstrass theorem. Series of real numbers, absolute convergence, tests of convergence for series of positive terms – comparison test, ratio test, root test; Leibniz test for convergence of alternating series.

Interior points, limit points, open sets, closed sets, bounded sets, connected sets, compact sets, completeness of R. Power series (of real variable), Taylor's series, radius and interval of convergence, term-wise differentiation and integration of power series

CSIR-NET Mathematical Sciences

Analysis:_Elementary set theory, finite, countable and uncountable sets, Real number system as a complete ordered field, Archimedean property, supremum, infimum. Sequences and series, convergence, limsup, liminf. Bolzano Weierstrass theorem, Heine Borel theorem. Continuity, uniform continuity, differentiability, mean value theorem. Sequences and series of functions, uniform convergence. Riemann sums and Riemann integral, Improper Integrals. Monotonic functions, types of discontinuity, functions of bounded variation, Lebesgue measure, Lebesgue integral. Functions of several variables, directional derivative, partial derivative, derivative as a linear transformation, inverse and implicit function theorems. Metric spaces, compactness, connectedness. Normed linear Spaces. Spaces of continuous functions as examples.

Praveen Chhikara

PRAVEEN CHHIKARA has been involved in teaching higher mathematics since 2012. He believes that the profession of teaching can act a big role in transforming the society towards positivity. Moreover it keeps life youthful in the company of young students. "*It gives me pleasure to be with students. It is a fun. They learn from me and so do I. These two converse processes make me bold and bolder day by day*," says Praveen Chhikara. He has a community of more than 8 thousand via teaching, social networking and his blogs. The community involves teachers and students pursuing their career at the prestigious institutions of the country.

Praveen Chhikara completed his master's degree in Maths from IIT Delhi. He is currently involved in an NGO "**Mathematical Community**", to contribute his skills in the development of mathematics education and education system at large.

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Real Analysis

- Roaster form: $\{1,2,3,\dots\}$

- Cardinality of a set := No. of elements in a set.
  e.g: $A=\{a,b,c\}$ → $|A|=3$
  Notation: $|A|$ → Cardinality of A.

- Singleton set: Cardinality is one.
  e.g: $\{3\}$ → Cardinality is 1.

★ $\{x\in\mathbb{R} : x^2=-1\} = \{\ \} = \phi$
  ϕ: Norwegian symbol
  group – Nicholas Bourbaki
  André Weil: The Apprentice of a Mathematician

- Finite set: Cardinality is finite.

★ $A\setminus B :\overset{\text{is defined as}}{=} \{x\in A : x\notin B\}$
  e.g: ① $\mathbb{R}\setminus\mathbb{Q}$ → set of all irrational numbers.
  ② $A=\{1,2,3,4\}$ $B=\{2,4,7,8\}$ $C=\{10,11,12\}$
  $A\setminus B=\{1,3\}$ $A\setminus C=\{1,2,3,4\}$

- Subset: A, B: set
  A is a subset of B i.e. $A\subseteq B \Rightarrow$ "$x\in A \Rightarrow x\in B$"

Venn diagram

★ $A=\{a_1,a_2,\dots,a_n\}$ is a set if $a_i\neq a_j$, for any i,j.
  $\{1,2,3\}=\{2,3,1\}$ → Order is immaterial!
  $\{1,1,2,3\}$ → Not allowed
  No repetition is allowed in sets.

⊛ In permutation, or means addition ~~and~~ means multiplication.

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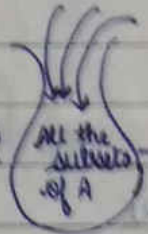
★ $A = \{a_1, a_2, \ldots, a_n\}$ → finite set
2 choices × 2 choices × ... × 2 choices $= 2^n$ → No. of subset of A.

★ ${}^nC_1$ → choose 1 element from n elements
${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$ —— ②
No one comes, 1 come, 2 comes, All comes
$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n$ —— ①
Put $x=1$ in ①, we get ②

⊛ $|A| = n < \infty$, then the subset of A is 2^n.

★ A: set
If $|A| = n$, then $|P(A)| = 2^n$.



→ Powerset of A denoted as P(A)

⊛ P(A) is never empty ($\because$ n has at least value 0 ⇒ $2^0 = 1$)

Q If $A = \{1, \{1,2\}, \phi, 3\}$, then which of the following is(are) true?
① $2 \in A$
② $\{1,2\} \subseteq A$ ($1 \in A$ but $2 \notin A$)
✓③ $\phi \subseteq A$ (always be true whatever be A)
✓④ $\{1,2\} \in A$
⑤ $\{1,2,3\} \subseteq A$ ($2 \notin A$)

⊛ empty set is a subset of every set

★ A, B: sets
$A \cup B = \{x : x \in A \text{ or } x \in B\}$
union

Q If $A \cup B = A$, then
(a) $A = B$ (not necessarily)
(b) $A \subseteq B$
✓(c) $B \subseteq A$

★ $A \cap B = \{x : x \in A \text{ and } x \in B\}$

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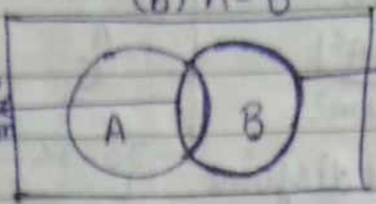
Q. If $A \cup B = A \cup C$ and $A \cap B = A \cap C$, then

(a) $A = B = C$ (b) $A = B$ (c) $A = C$ (d) $B = C$

Inside part of C (if we take less then

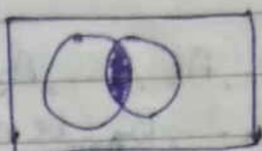


→ Outside part of C (if we take more then $A \cup B \neq A \cup C$.)

⊛ $A \cap B = A$, then $A \subseteq B$

★ A, B: finite sets

$|A \cup B| = |A| + |B| - |A \cap B|$



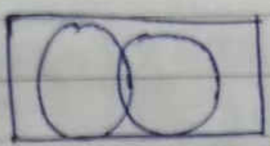
Q. A survey show that 63% of Americans like apples and 76% like cheese. If x% like both, then find x.

Sol^n $|A| = 63$ $|C| = 76$ $|A \cap C| = ?$

$|A \cap C| = |A| + |C| - |A \cup C| = 63 + 76 - (76 \,\&\, 100) = 39$ to 63

$\therefore 39 \leq |A \cap C| \leq 63$

★



$|A| + |B| \geq |U|$

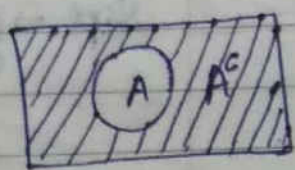
$|A| + |B| \leq |U|$

⊛ $\max\{|A|, |B|\} \leq |A \cup B| \leq \min\{|A| + |B|, |U|\}$

⊛ $|\phi|$ $0 \leq |A \cap B| \leq \min\{|A|, |B|\}$

• Complement of a set: $A^c = U \setminus A$

$A \cup A^c = U$



Q. In a battle, 70% of the combatants lose one eye, 80% lose an ear, 75% lose a leg and 85% lose an arm. If x% lose all the four limbs. Find the minimum value of x.

Sol^n $x = |E_y \cap E_a \cap L \cap A|$

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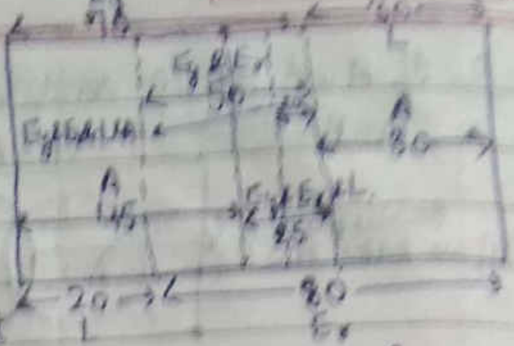
⊛ <u>De Morgan's Laws</u>: $(A\cup B)^c = A^c \cap B^c$
$(A\cap B)^c = A^c \cup B^c$



$|A| = |U| - |A^c|$

$x = |U| - |(E_g \cap E_n \cap L \cap A)^c|$

$= 100 - |E_g^c \cup E_n^c \cup L^c \cup A^c|$

Minimum when it is maximum?

when they are disjoint

$|E_g^c| + |E_n^c| + |L^c| + |A^c| = 30 + 20 + 25 + 15 = 90$

$\therefore x = 100 - 90 = 10$

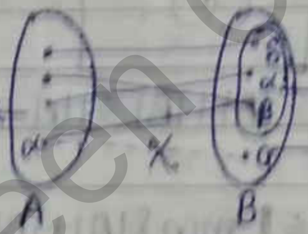
Q Let $A_1, A_2, \ldots, A_{30}$ be 30 sets, each with 5 elements, and $B_1, B_2, \ldots, B_n$ be n sets, each with 3 elements. Suppose $\bigcup_{i=1}^{30} A_i = S = \bigcup_{j=1}^{n} B_j$. If each element of S is a member of exactly 10 A_i's & each element of S is a member of exactly 9 B_j's, then $n = ?$

Solⁿ: $|S| = \frac{150}{10} = \frac{3n}{9} \Rightarrow n = 45$

$\alpha \in A_1 \Rightarrow \alpha \in S \ldots$

$\alpha \in A_i \rightarrow 10\ A_i$'s

- <u>Function</u>:



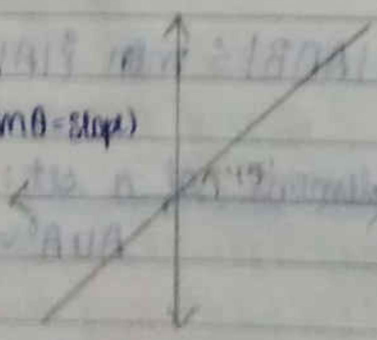
Range ⊆ Co-domain

- <u>Graphs</u>:

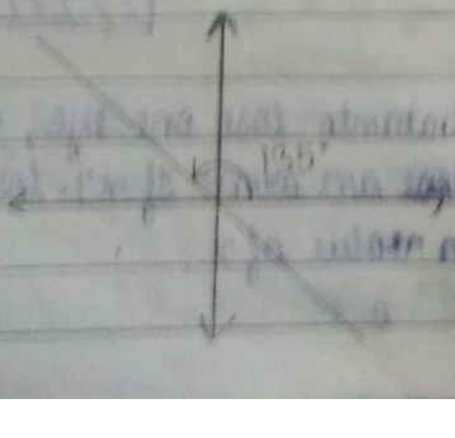
1) $y = x \rightarrow$ Identity function

Slope $\rightarrow \frac{dy}{dx} = 1 \rightarrow \tan\theta = 1$ ($\because \tan\theta$ = slope)

$\rightarrow \theta = 45°$



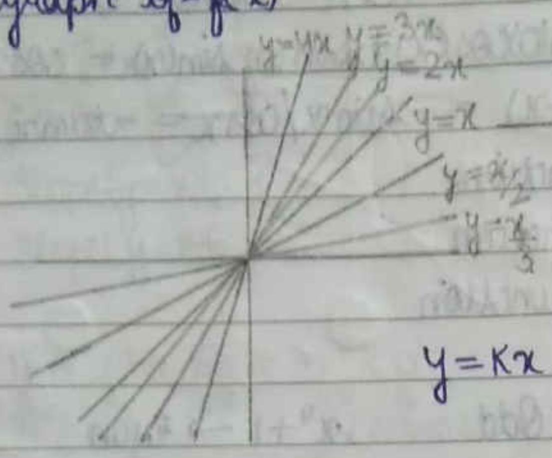
2)



$y = -x$

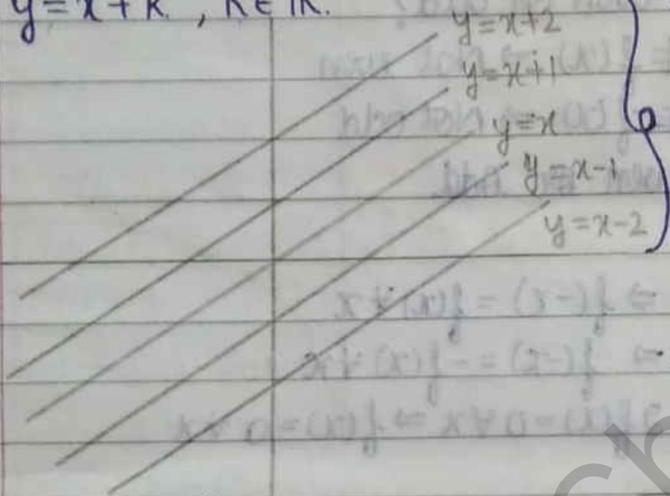
$f(-x) = f(x) \rightarrow$ Even func^n
$f(-x) = -f(x) \rightarrow$ Odd func^n

★ Graph of $f(x)$

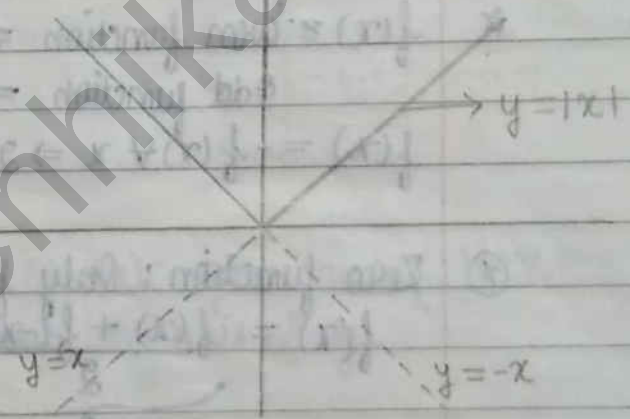
3) 

$y = Kx$, $K \in \mathbb{R}$

4) $y = x + K$, $K \in \mathbb{R}$.



All are parallel ~~and~~ as slope are same

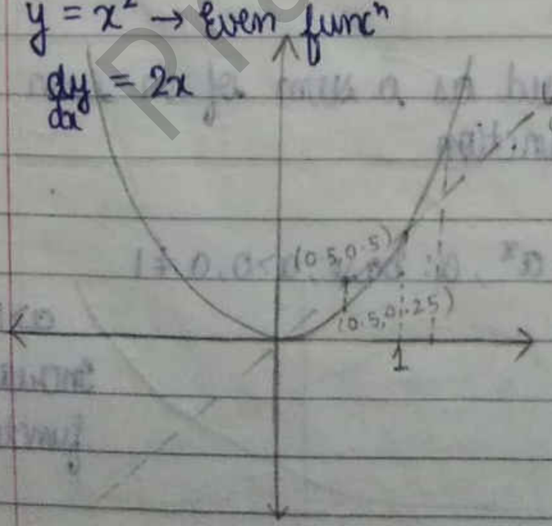


5) $y = |x| = \begin{cases} x : x \geq 0 \\ (-x) : x < 0 \end{cases}$
+ve -ve

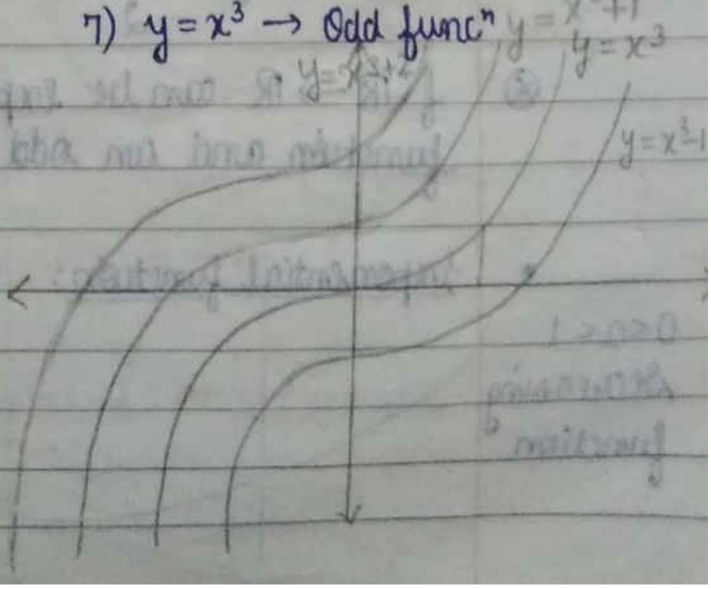
★ 0 to 1 $\rightarrow (0.5)^1 = 0.5 \quad (0.5)^2 = 0.25 \quad (0.5)^3 = 0.0125$

6) $y = x^2 \rightarrow$ Even func^n

$\frac{dy}{dx} = 2x$



7) $y = x^3 \rightarrow$ Odd func^n



★ $\sin(-x) = \sin(0-x) = \sin(0)\cos(x) - \cos(0)\sin(x) = -\sin x$

$\cos(-x) = \cos(0-x) = \cos(0)\cos(x) + \sin(0)\sin(x) = \cos x$

$\tan(-x) = \sin(-x)/\cos(-x) = -\sin x/\cos x = -\tan x$

⊛ Sine function: Odd function
Cosine function: Even function
Tangent function: Odd function

★ $x^4 \rightarrow$ Even $\qquad x^5 \rightarrow$ Odd $\qquad x^4+1 \rightarrow$ Even

$f(x) = x^4 + x^3 \rightarrow$ Even or odd?

$f(-x) = -x^4 - x^3 \neq f(x) \rightarrow$ Not even

$\neq -f(x) \Rightarrow$ Not odd

$\therefore f(x)$ is neither even or odd

★ $f(x)$: Even function $\Rightarrow f(-x) = f(x)\ \forall x$

Odd function $\Rightarrow f(-x) = -f(x)\ \forall x$

$f(x) = -f(x)\ \forall x \Rightarrow 2f(x) = 0\ \forall x \Rightarrow f(x) = 0\ \forall x$

⊛ Zero function: Only function which is even as well as odd.

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{g(x)} + \underbrace{\frac{f(x)-f(-x)}{2}}_{h(x)}$$

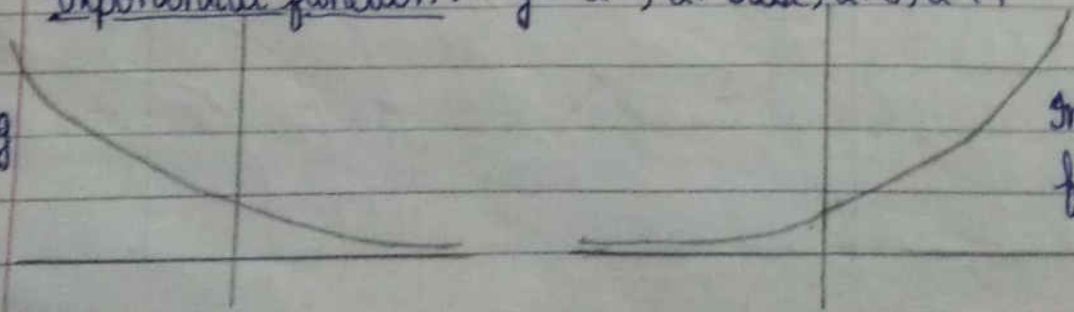
$g(-x) = \frac{f(-x)+f(x)}{2} = g(x) \Rightarrow g(x)$ is even

$h(-x) = \frac{f(-x)-f(x)}{2} = -h(x) \Rightarrow h(x)$ is odd

⊛ $f: \mathbb{R} \rightarrow \mathbb{R}$ can be expressed as a sum of an even function and an odd function

• <u>Exponential function</u>: $y = a^x$, a: base, $a>0, a \neq 1$

$0<a<1$
Decreasing function



$a>1$
Increasing function

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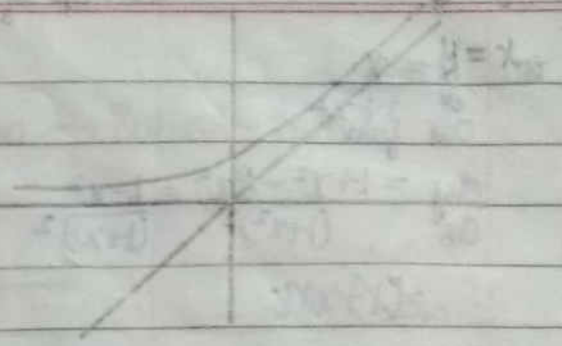
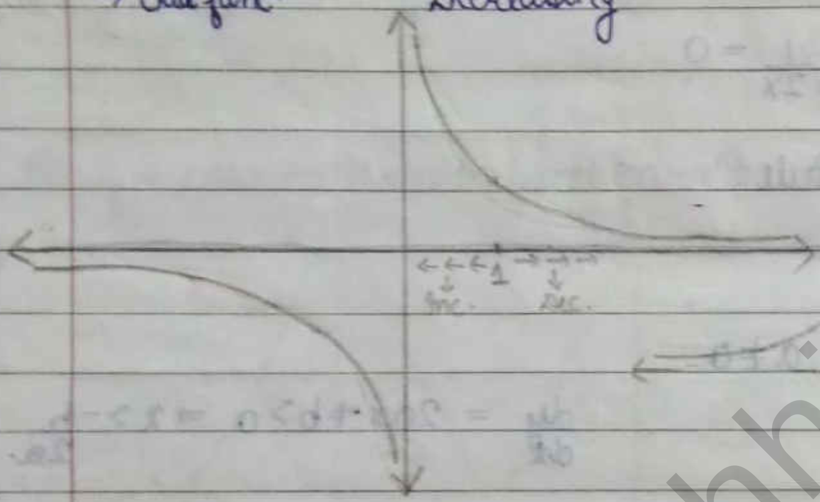
$\left[\frac{\infty}{\infty}\right] \rightarrow$ Indeterminate form

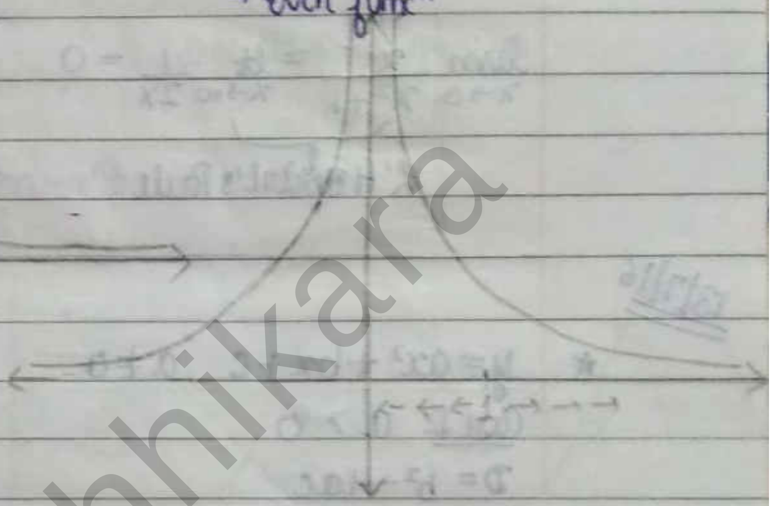
★ $A = \{(x, y) : y = x\}$
$B = \{(x, y) : y = e^x\}$
$A \cap B = ? = \phi$
Slope$(y = x)$: 1
Slope$(y = e^x)$: e^x



★① $y = \frac{1}{x} \Rightarrow \frac{dy}{dx} = -\frac{1}{x^2} < 0$ ⇓ Decreasing
↳ Odd func^n

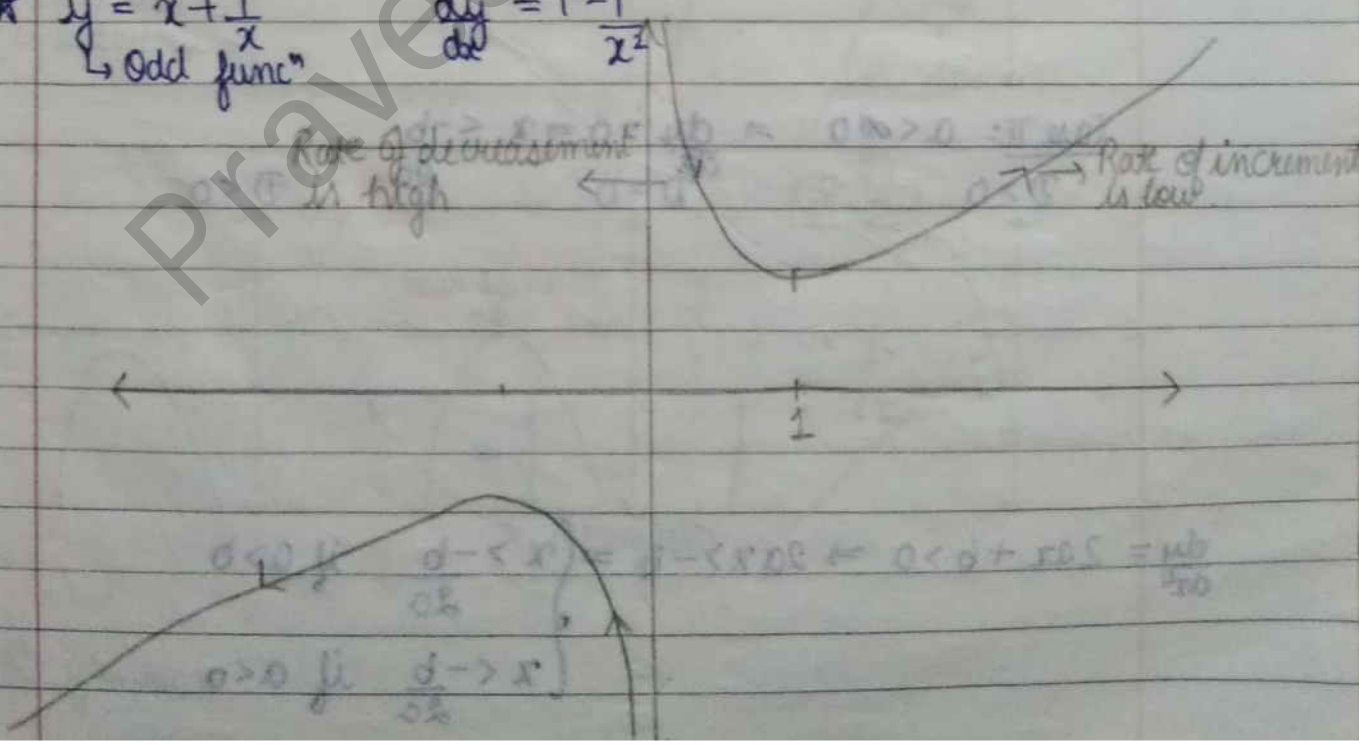
② $y = \frac{1}{x^2} \Rightarrow \frac{dy}{dx} = -\frac{2}{x^3}$
↳ Even func^n





⊛ Odd function: Symmetric in opposite quardants (1 & 3 are same, 2 & 4 are same)
Even function: symmetric about the axis of y.

★ $y = x + \frac{1}{x}$ $\quad \frac{dy}{dx} = 1 - \frac{1}{x^2}$
↳ Odd func^n



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$\alpha < \beta$

$b\alpha < b\beta$ if $b > 0$

$b\alpha > b\beta$ if $b < 0$

★ $y = \dfrac{x}{1+x^2}$

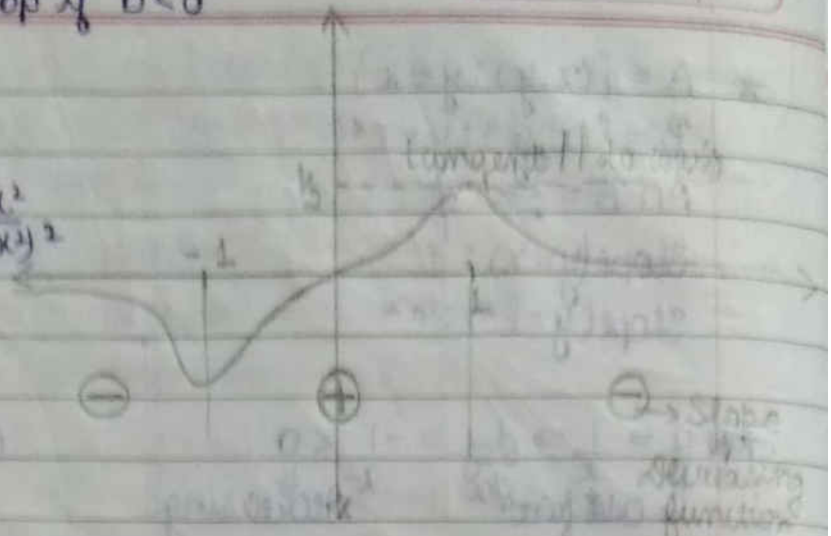
Odd fun^n

$\dfrac{dy}{dx} = \dfrac{1+x^2-2x^2}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2}$

$\cancel{(1+x)x}$

Slope at 0 $= \left.\dfrac{dy}{dx}\right|_{x=0} = 1$

$\left.\dfrac{dy}{dx}\right|_{x=1} = 0$



$\underbrace{\lim_{x\to 0} \dfrac{x}{x^2+1} = \operatorname{lt}_{x\to\infty} \dfrac{1}{2x}}_{\text{L'Hôpital's Rule}} = 0$

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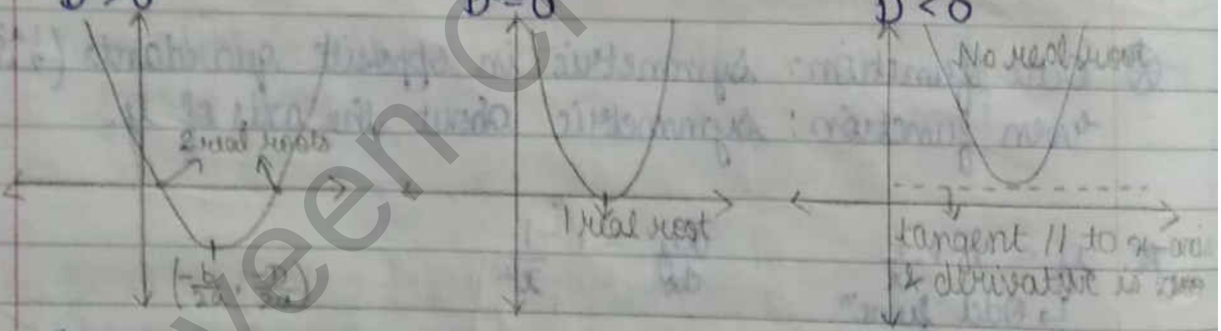
★ $y = ax^2 + bx + c$, $a \neq 0$

Case I: $a > 0$ $\qquad \dfrac{dy}{dx} = 2ax + b > 0 \Rightarrow x > -\dfrac{b}{2a}$

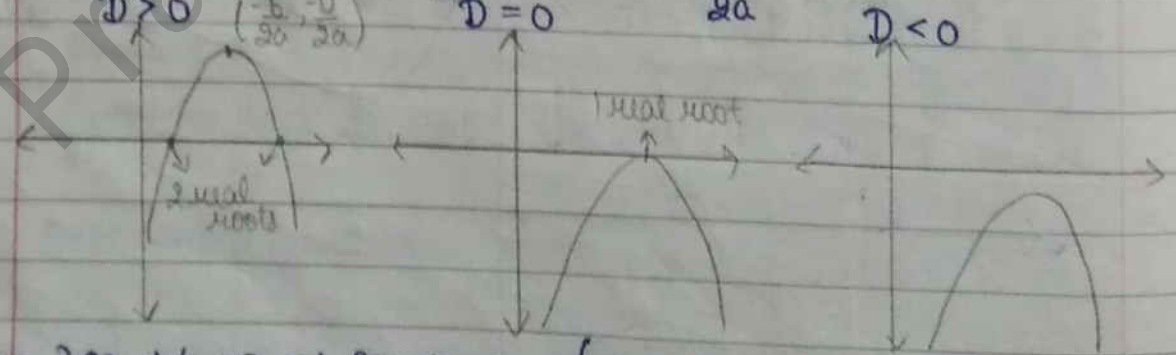
$D = b^2 - 4ac$

$D > 0$ $\qquad D = 0$ $\qquad D < 0$



Case II: $a < 0$ $\Rightarrow \dfrac{dy}{dx} > 0 \Rightarrow x < -\dfrac{b}{2a}$

$D > 0$ $\left(-\frac{b}{2a}, -\frac{D}{2a}\right)$ $\qquad D = 0$ $\qquad D < 0$



$$\frac{dy}{dx} = 2ax + b > 0 \Rightarrow 2ax > -b \Rightarrow \begin{cases} x > -\dfrac{b}{2a} & \text{if } a > 0 \\ x < -\dfrac{b}{2a} & \text{if } a < 0 \end{cases}$$

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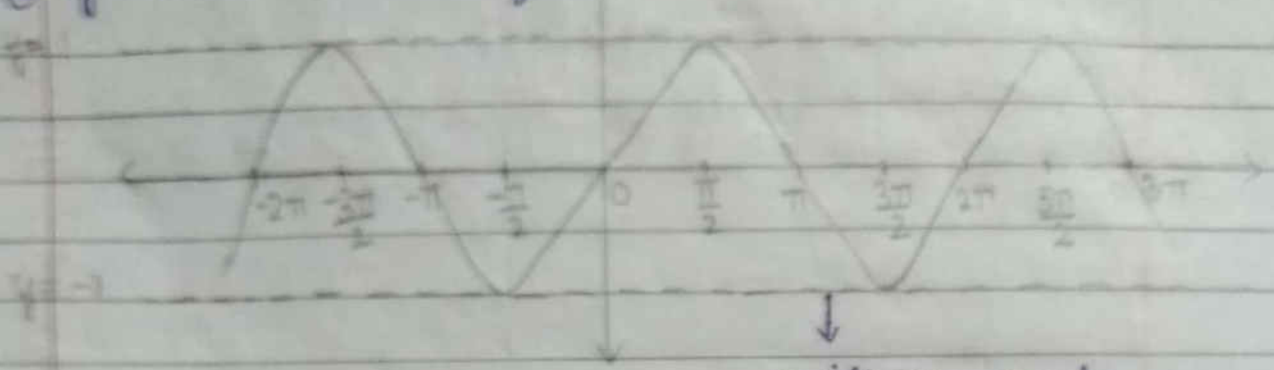
- <u>Trigonometrical functions</u>:

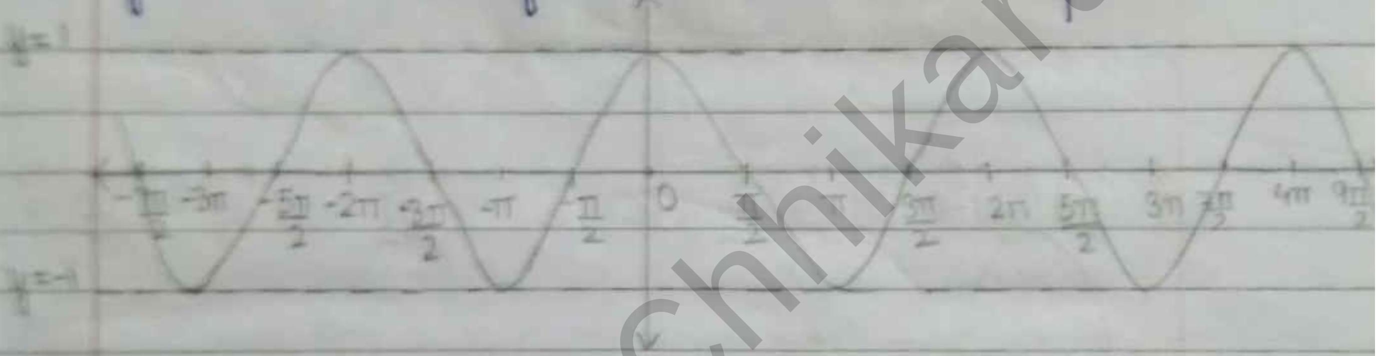
① $y = \sin x \rightarrow$ Periodic function $\rightarrow$ Period $= 2\pi$



$-1 \le \sin x \le 1$

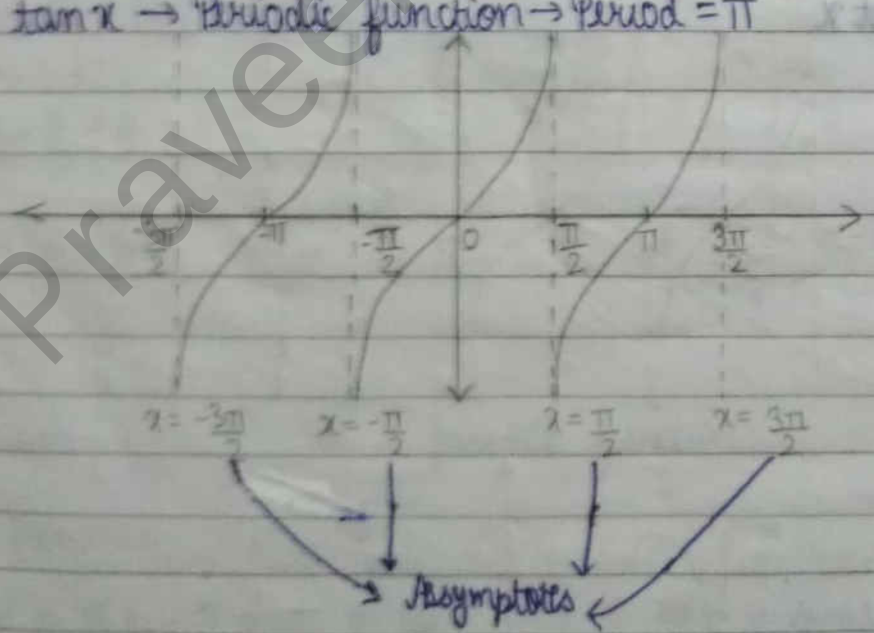
Oscillatory graphs

② $y = \cos x \rightarrow$ Periodic function $\rightarrow$ Period $= 2\pi$



$-1 \le \cos x \le 1$

③ $y = \tan x \rightarrow$ Periodic function $\rightarrow$ Period $= \pi$



④ $y = \sec x$

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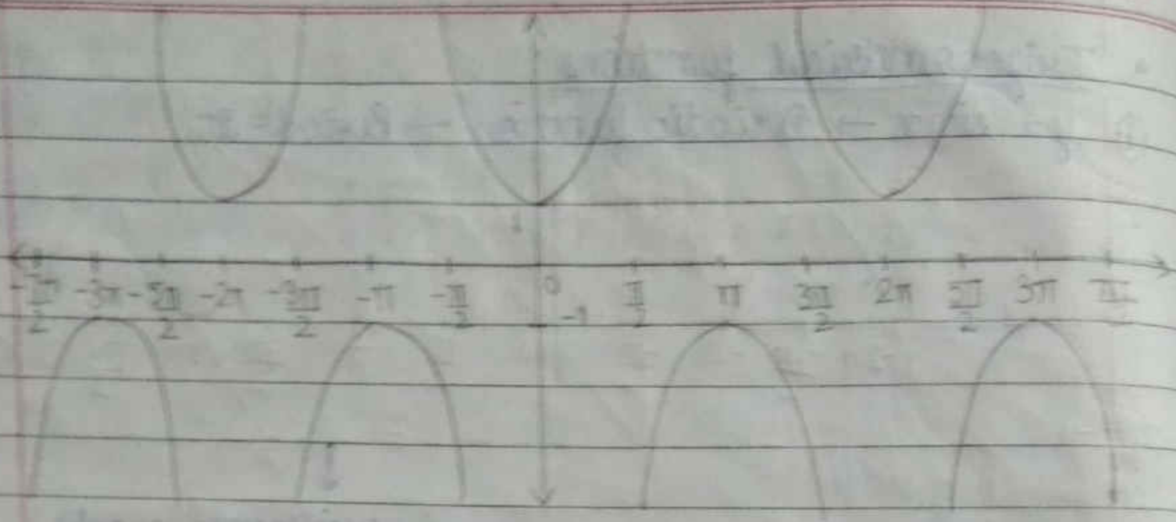
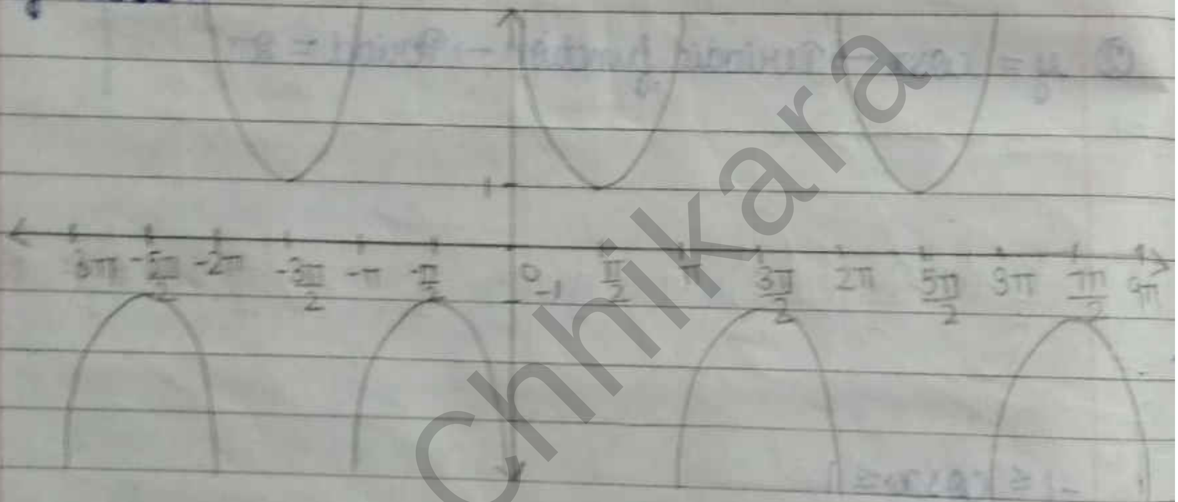
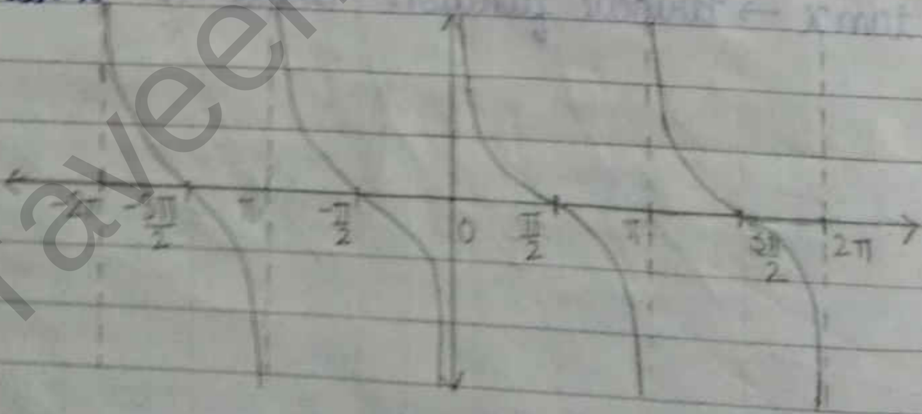


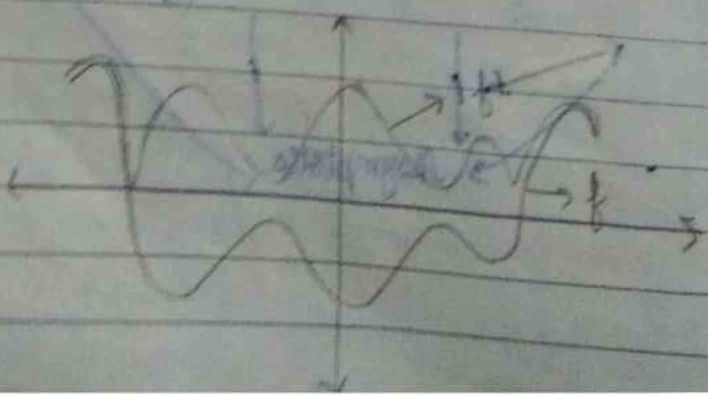
⑤ $y = \operatorname{cosec} x$



⑥ $y = \cot x$



* $|f|$



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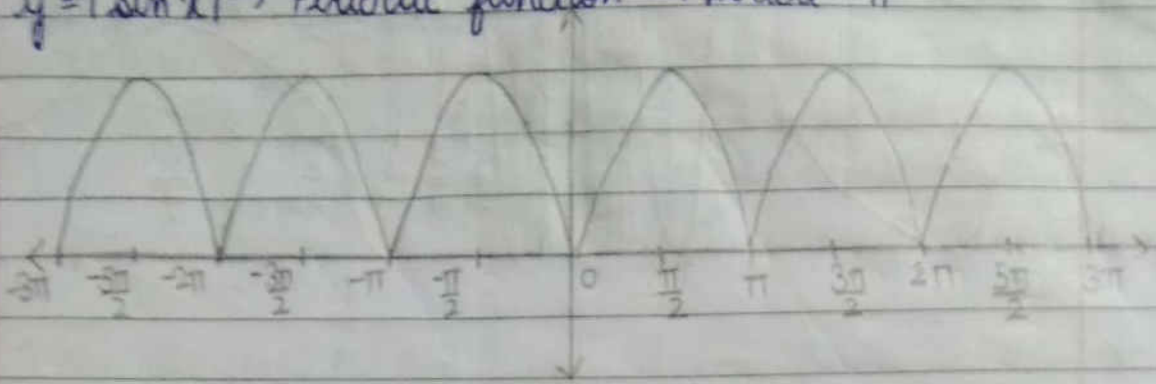
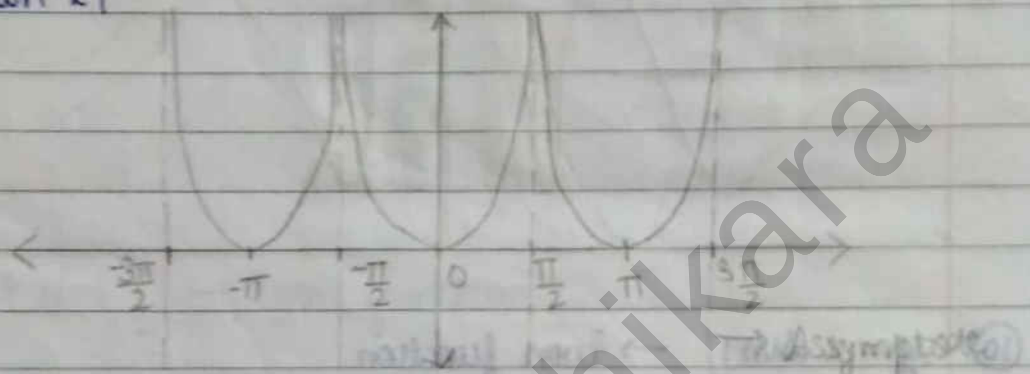
⑦ $y = |\sin x| \rightarrow$ Periodic function $\rightarrow$ Period $= \pi$



⑧ $y = |\tan x|$



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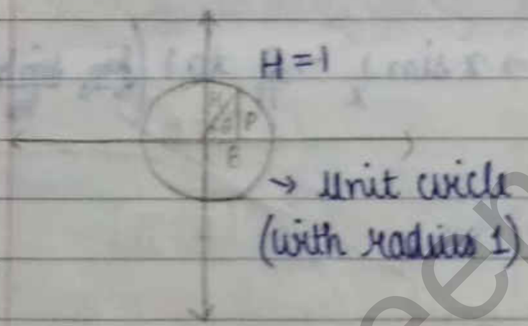

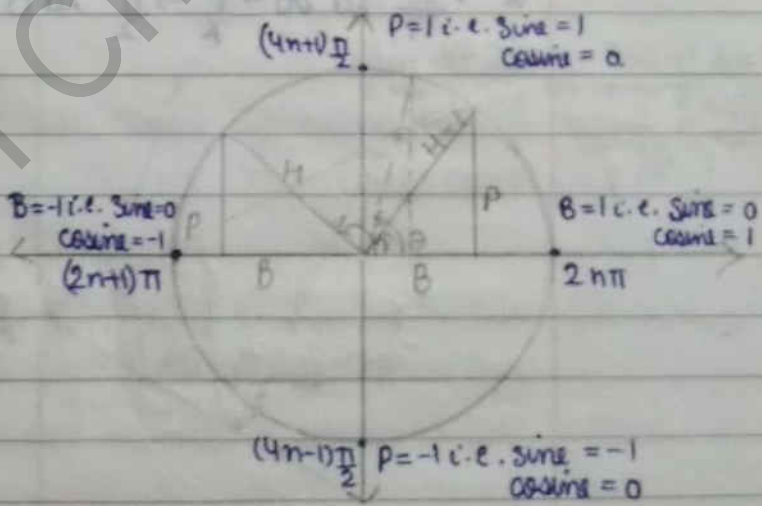

$\sin\theta = \frac{P}{H} = P$

$\cos\theta = \frac{B}{H} = B$

⑨ $y = \sin\left(\frac{1}{x}\right) \rightarrow$ Not a periodic function

Odd function.

$\lim \frac{V}{x} \neq 1$ $\quad x \rightarrow \frac{2}{\pi}$ to $\infty \rightarrow \frac{1}{x} \rightarrow \underbrace{\frac{\pi}{2} \text{ to } 0}_{\text{Ist quardant}} \rightarrow \sin\frac{1}{x} \rightarrow 1$ to 0

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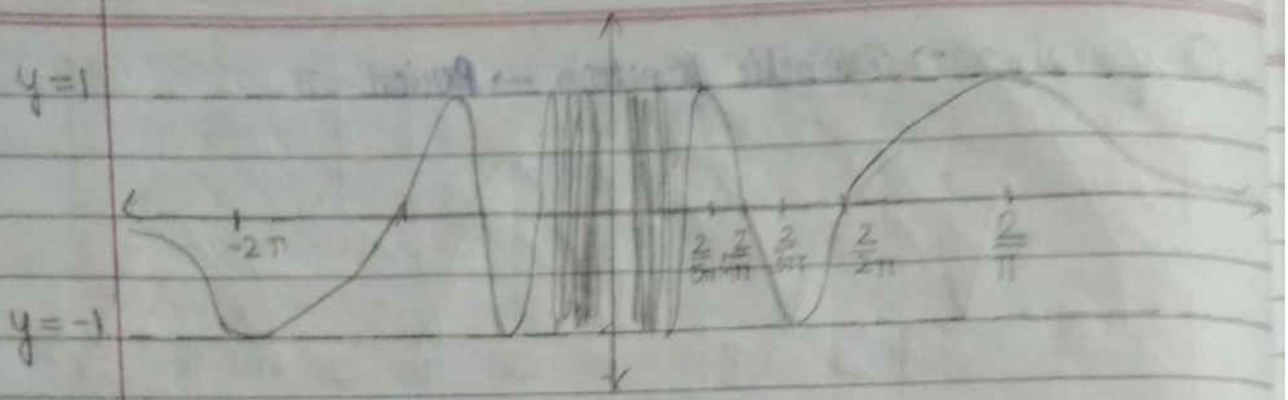
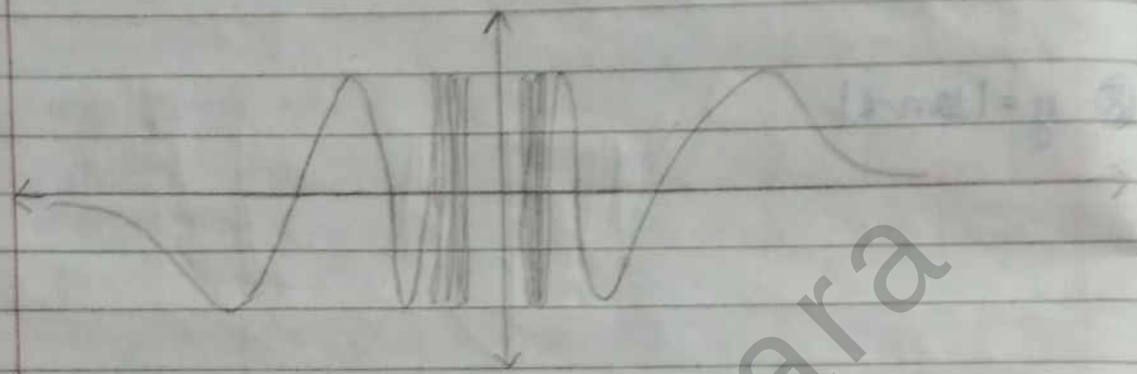
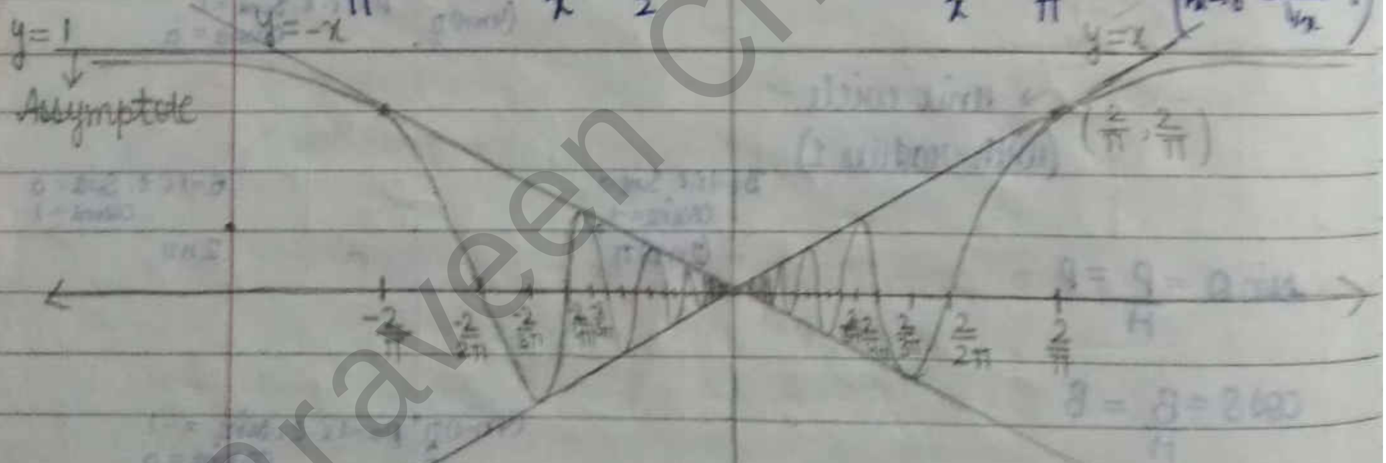
$\frac{2}{\pi} < 1 \quad \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$





⑩ $y = x \sin\frac{1}{x}$ → Even function

$x \to \frac{2}{\pi}$ to $\infty \Rightarrow \frac{1}{x} \to \frac{\pi}{2}$ to $0 \Rightarrow x \sin\frac{1}{x} \to \frac{2}{\pi}$ to 1 $\left(\lim_{1/x \to 0} \frac{\sin 1/x}{1/x} = 1\right)$



⑪ $y = x^2 \sin\frac{1}{x}$ → Odd function

$x \to \frac{2}{\pi}$ to $\infty \Rightarrow \frac{1}{x} \to \frac{\pi}{2}$ to $0 \to x^2 \sin\frac{1}{x} \to \frac{4}{\pi^2}$ to ∞

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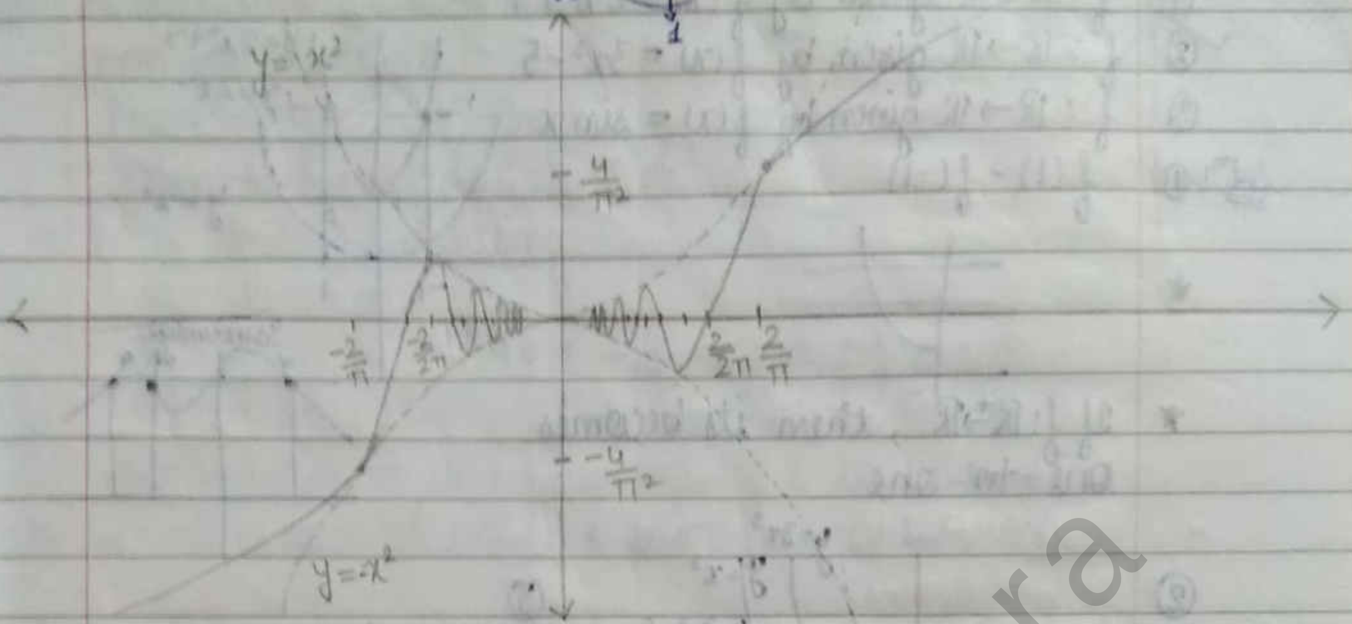
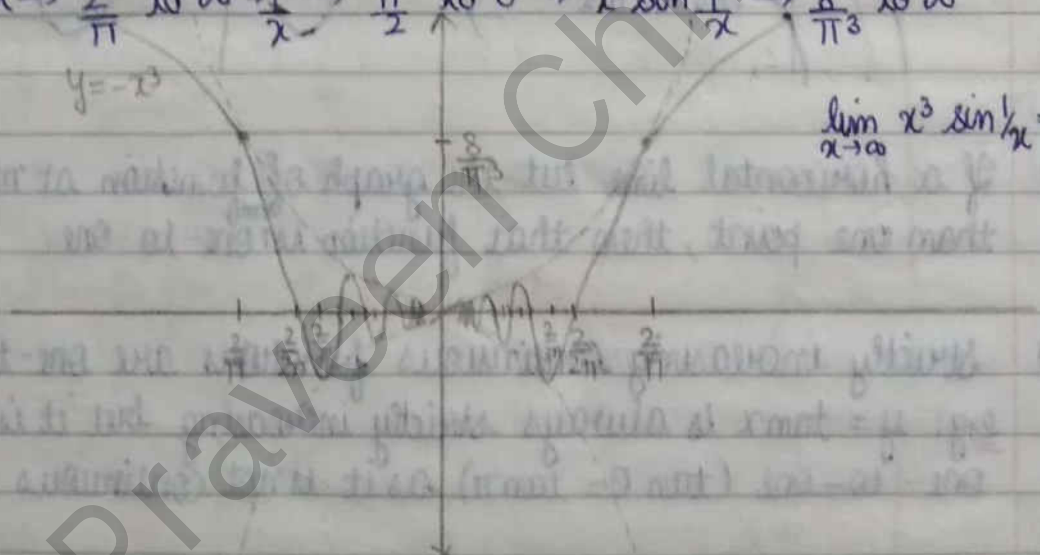
$\lim_{x\to\infty} x^2 \sin\frac{1}{x} = \lim_{x\to\infty} \underset{\downarrow}{x}\;\underset{\downarrow}{\frac{\sin 1/x}{1/x}} = \infty$
$\qquad\qquad\qquad\qquad\infty \qquad\; 1$



⑫ $y = x^3 \sin\frac{1}{x} \rightarrow$ Even function

$x \rightarrow \frac{2}{\pi}$ to $\infty \Rightarrow \frac{1}{x} \rightarrow \frac{\pi}{2}$ to $0 \Rightarrow x^3 \sin\frac{1}{x} \rightarrow \frac{8}{\pi^3}$ to ∞



$\lim_{x\to\infty} x^3 \sin 1/x = \lim_{x\to\infty} \underset{\downarrow \infty}{x^2} \underset{\downarrow 1}{\frac{\sin 1/x}{1/x}}$
$= \infty$

- <u>One-to-One functions</u>: $f: A \rightarrow B$

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

or $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$P \Rightarrow Q$
or $\cancel{Q} \Rightarrow \cancel{P}$] equivalent.

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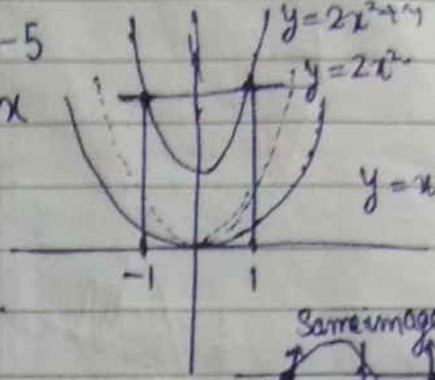
Q Which of the following are one-to-one?

① $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x^2 + 4$

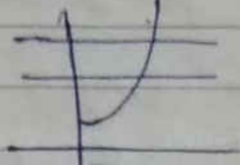
② $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x^3 - 5$

③ $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin x$

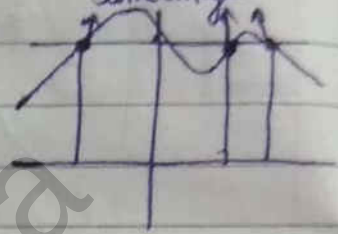
Solⁿ: ① $f(1) = f(-1)$



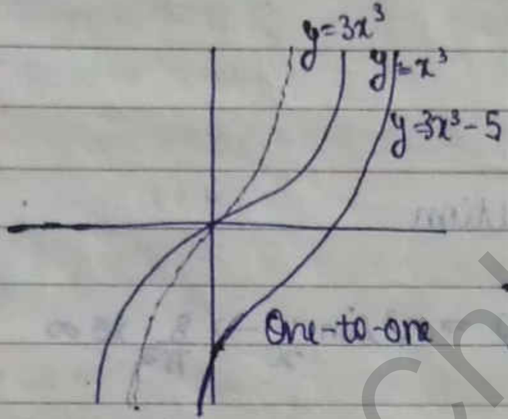
★



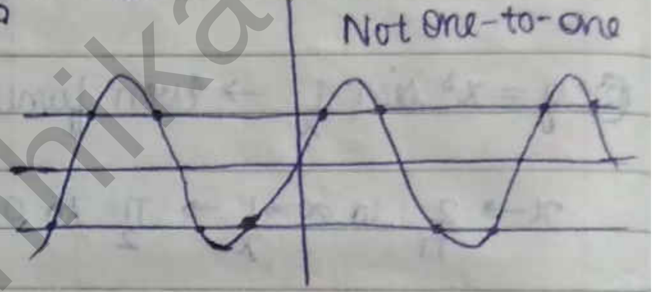
If $f: \mathbb{R}^+ \to \mathbb{R}$, then its becomes one-to-one



②



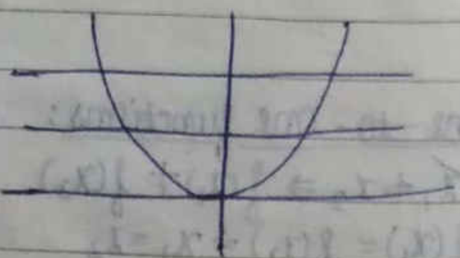
③



⊛ If a horizontal line cut the graph of function at more than one point, then that function is not one-to-one.

⊛ Strictly increasing continuous functions are one-to-one
e.g: $y = \tan x$ is always strictly increasing but it is not one-to-one ($\tan 0 = \tan \pi$) as it is not continuous.

★ $f: \mathbb{R} \to \mathbb{R}$
$f(x) = x^n$



Case I: n: even $\to$ Not one-to-one

Case II: n: odd

$f'(x) = n x^{\overbrace{n-1}^{\text{even}}} \geq 0$

strictly increasing & continuous $\Rightarrow$ one-to-one

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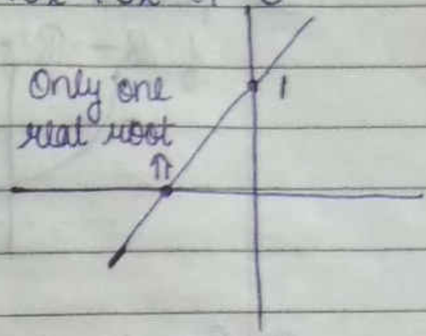
Non real roots occur in conjugate pairs

Q Find the number of real root of $9x^9+7x^7+5x^5+3x^3+1=0$

Solⁿ: At least one real root as max. # of non real roots $=8$.



$f(x)=9x^9+7x^7+5x^5+3x^3+1$

$f'(x)=81x^8+49x^6+25x^4+9x^2 \geq 0$

$\Rightarrow$ f is strictly increasing

$\lim\limits_{x\to\infty} f(x)=\infty$

$\lim\limits_{x\to-\infty} f(x)=\underset{\downarrow\,-\infty}{x^9}\left(\underset{\downarrow\,9}{9+\frac{7}{x^2}+\frac{5}{x^4}+\frac{3}{x^6}+\frac{1}{x^9}}\right)=-\infty$

The function is strictly increasing & continuous.

$\Rightarrow$ It cut x-axis at only one point $\Rightarrow$ One real root.

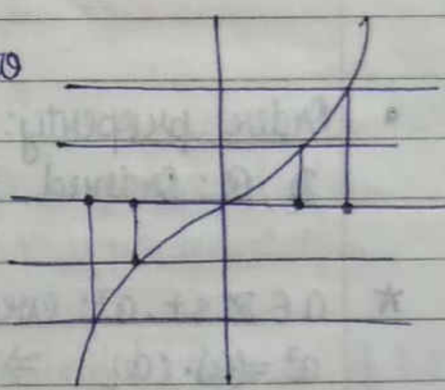
- **Onto functions:** $f: A\to B$

  Range = Codomain

e.g.: ① $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\sin x$

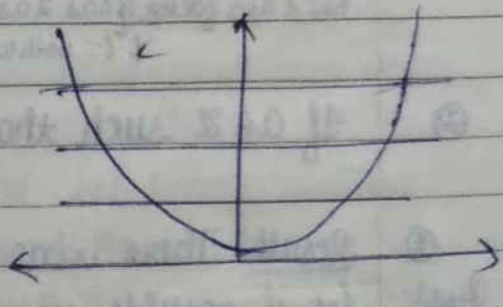
Range$(f)=[-1,1] \Rightarrow$ Not onto

② $f:\mathbb{R}\to[-1,1]$, $f(x)=\sin x \Rightarrow$ Onto



③ ✱ $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^3 \to$ ~~Not~~ onto

④ $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^2\to$ Not onto



⑤ $f:\mathbb{R}\to\mathbb{R}^+$, $f(x)=x^2 \to$ Onto

✱ If a horizontal line cuts the graph of function at least one point, then that funcⁿ is onto.

✱ ~~Onto to one~~ Onto: Each $\underset{\text{spring codomain}}{\underline{\text{horizontal line}}}$ cuts the graph at at least one

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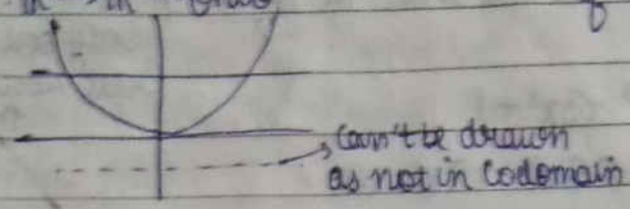
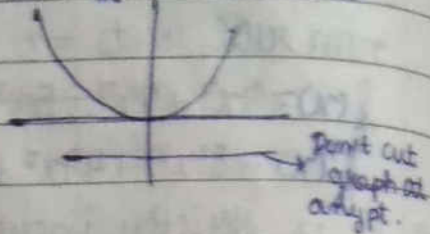
point.

$f: \mathbb{R} \to \mathbb{R}^+ \to$ onto



$f: \mathbb{R} \to \mathbb{R} \to$ Not onto



⊛ <u>One-to-one</u>: Each horizontal line cuts the graph at not more than one pt.

$f: \mathbb{R} \to \mathbb{R} \to$ Not one-to-one

- <u>Fields</u>: $\mathbb{F} \subseteq \mathbb{R}$

$$\left.\begin{array}{l} a, b \Rightarrow a+b \in \mathbb{F} \\ \Downarrow \; a-b \in \mathbb{F} \\ \Downarrow \; a \times b \in \mathbb{F} \\ a \div b \in \mathbb{F}, b \neq 0 \end{array}\right\} \to \mathbb{F}: \text{field}$$

$\mathbb{N}$ ✗
$\mathbb{R}$ ✓
$\mathbb{Z}$ ✗
$\mathbb{Q}$ ✓
$\mathbb{R}\setminus\mathbb{Q}$ ✗

$(1-\sqrt{2})+\sqrt{2} = 1 \notin \mathbb{R}\setminus\mathbb{Q}$

- <u>Order property</u>: $\left.\begin{array}{l} \mathbb{R} \\ a, b \in \mathbb{R}, a \neq b \end{array}\right] \Rightarrow a<b$ or $b<a$

$\mathbb{Q}, \mathbb{R}$: Ordered fields (has order property)

★ $a \in \mathbb{Z}$ s.t. a^2: even

$a^2 = (a)\cdot(a) \Rightarrow a$ is even.

has 2 as a factor, has 2 as a factor (∵ both are same)

⊛ If $a \in \mathbb{Z}$ such that a^2 is even, then a must be even.

⊛ <u>Result</u>: There is no rational number x s.t. $x^2 = 2$

<u>Proof</u>: Let if possible, $x^2 = 2$, $x \in \mathbb{Q}$

$x = p/q$; $p, q \in \mathbb{Z}, q \neq 0$, g.c.d $(p,q) = 1$

$x^2 = 2 \Rightarrow \left(\frac{p}{q}\right)^2 = 2 \Rightarrow \frac{p^2}{q^2} = 2 \Rightarrow p^2 = 2q^2$ — ①

$\Rightarrow p^2$ is even $\Rightarrow p$ is even — Ⓐ

$\Rightarrow p = 2m, m \in \mathbb{Z}$

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"Completeness" $\rightarrow$ helps in differentiating b/w $\mathbb{R}$ & $\mathbb{Q}$

Put $p = 2m$ in ①

$4m^2 = 2q^2 \Rightarrow q^2 = 2m^2 \Rightarrow q^2$ is even $\Rightarrow q$ is even — Ⓑ

Ⓐ & Ⓑ $\rightarrow \underset{\text{g.c.d}(p,q)=2}{}$ $\Rightarrow\!\Leftarrow$ as g.c.d.$(p, q) = 1$

⊛ $\mathbb{Q} \subsetneq \mathbb{R}$ (By above result)

★ If we say that $\mathbb{R}$ is an ordered field, is it a complete characterization of $\mathbb{R}$? No, b/c $\mathbb{Q}$ is also ordered field.
We need a property that distinguishes $\mathbb{Q}$ & $\mathbb{R}$ $\rightarrow$ "Completeness"

★ $A \subseteq \mathbb{R}$, when to call a set bounded above?

A
b $\leftarrow$ A is bounded above by 'b'

- Definition: $A \neq \phi$, $A \subseteq \mathbb{R}$, A is s.t.b. a bounded above set if $\exists$ a 'b' $\in \mathbb{R}$ s.t. $a \leq b \ \forall \ a \in A$
  b: upper bound of A.

Q Give some upper bounds of
① $S_1 = \{x \in \mathbb{R} : x^2 \leq 4\} \rightarrow [-2, 2]$: 2 is an upper bound of S_1
② $S_2 = \{\sin x : x \in \mathbb{R}\} \rightarrow [-1, 1]$: 1 is an upper bound of S_2

- Definition: $A \neq \phi$, $A \subseteq \mathbb{R}$, A is s.t.b. a bounded below set if $\exists$ a 'b' $\in \mathbb{R}$ s.t. $a \geq b \ \forall \ a \in A$
  b: lower bound of A.

Q How many upper bounds and lower bounds does a bounded set have?
Solⁿ: If b: U.B, then $b+1, b+2, b+3, \ldots$ all are U.B.s.
If b: L.B., then $b-1, b-2, b-3, \ldots$ all are L.B.s.

⊛ Once one upper bound is known, we can find infinitely many upper bounds and similarly, once one lower bound is known, we can find infinitely many lower bounds.

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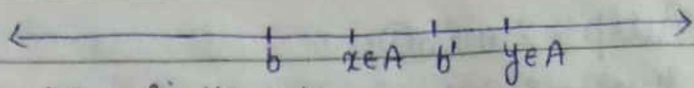
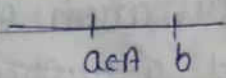
Peano
Terence Tau] $\rightarrow \mathbb{N} = \{0,1,2,\dots\}$

★ $A \subseteq \mathbb{R}, A \neq \phi$



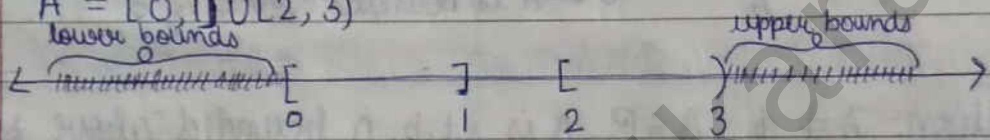
Unbounded above ⇒ Not a finite set
Absence of Upper bounds
Mean? ~~how~~ No number, however big, cannot be an upper bound of A.
For any $b \in \mathbb{R}$, $\exists\, a \in A$ s.t. $b < a$

★ $A \subseteq \mathbb{R}, A \neq \phi$
Absence of Lower bounds] Mean?



For any $b \in \mathbb{R}$, $\exists\, a \in A$ s.t. $b > a$

★ $A = [0,1] \cup [2,3)$



The smallest upper bound is significant ~~here~~ here.
How to define it?

- $A \neq \phi, A \subseteq \mathbb{R}$, s: least upper bound of A (Supremum)

① s is an upper bound of A
② If b is an upper bound of A, then $s \leq b$

- $A \neq \phi, A \subseteq \mathbb{R}$, t: greatest lower bound of A (Infimum)

① t is a lower bound of A
② If b is a lower bound of A, then $b \leq t$.

★ $A \neq \phi, A \subseteq \mathbb{R}$
A: bounded above & bounded below] → Bounded sets

- Unbounded sets: The set which is not bounded

Eg: $\mathbb{N} \rightarrow \{1,2,\dots\} \rightarrow$ bounded below (lower bound is 1) and not bounded above] ⇒ Not bounded

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Stadium → Stadia   Radium → Radia
Supremum → Suprema

15/7/16

Q How many suprema for a bounded above sets (non-empty) exist?

Solⁿ Claim: Unique supremum

$A \neq \emptyset,\ A \subseteq \mathbb{R}$

s_1, s_2 : suprema of A

$a \leq s_1\ \forall a \in A$ $a \leq s_2\ \forall a \in A$

$\left.\begin{array}{l} s_1 : \text{Sup},\ s_2 : \text{UB} \Rightarrow s_1 \leq s_2 \\ s_2 : \text{Sup},\ s_1 : \text{UB} \Rightarrow s_2 \leq s_1 \end{array}\right] \Rightarrow s_1 = s_2$

⊛ If supremum exists, then it is unique.

- Examples:

① $A = [0, 1)$

1 is the L.U.B. ⇐ $a \leq 1\ \forall a \in A \Rightarrow 1$ is an U.B. of A
Pick any no. < 1 : can't be an U.B. of A

infinitely many elements

So, $\sup A = 1 \notin A$

② $B = [0, 1]$

$\sup B = 1 \in B$

⊛ The supremum of a set may not belong to the set.

- Supremum & Maximal element
May be outside the set   Must be inside the set, if it exists

⊛ If a maximal element exists, it must be the supremum
Supremum $\not\Rightarrow$ Maximal element.

Q Find the supremum & infimum (if they exists).

① $[a, b)$ Maximum ✗

$\text{Sup} = b \notin A$ $\text{Inf} = a \in A$ (a) ← Minimum ✓

② $(a, b]$ (b) ← Maximum ✓

$\text{Sup.} = b \in A$ $\text{Inf} = a \notin A$ Minimum ✗

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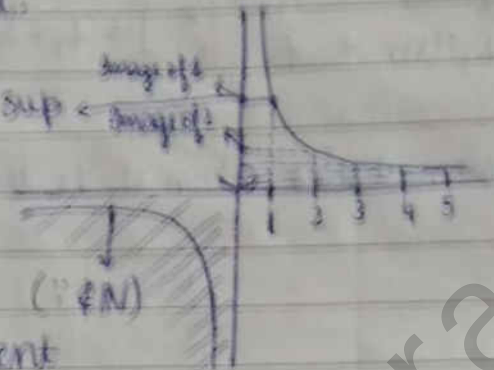
③ $\{1,2,3,4\}$

Sup $= 4 \in A$ — Maximum element Inf $= 1 \in A$ — Minimum element

* A finite ~~&~~ nonempty set has a maximum as well as minimum element.

Imp ④ $\{\frac{1}{n} : n \in \mathbb{N}\}$



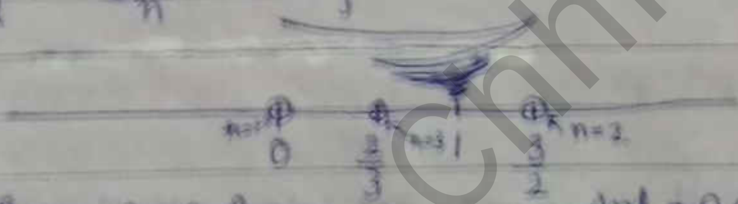
$f(x) = \frac{1}{x}$

Sup $= 1 \in A$

Maximum element

Inf. $= 0 \notin A \Rightarrow$ Minimum element doesn't exist.

⑤ $\{1 + \frac{(-1)^n}{n} : n \in \mathbb{N}\}$



Sup $= \frac{3}{2} \in A$ — Maximum element Inf $= 0 \in A$ — Minimum element

⑥ $\{1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{2^2}, \ldots\}$

Inf $= 1 \in A$

Minimum element

Last element $= 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots = \frac{1}{1-\frac{1}{2}} = 2$

geometric series

$a + ar + ar^2 + \ldots = \frac{a}{1-r} \quad -1 \le r \le 1$

Sup $= 2 \notin A$

Maximum element doesn't exist.

⑦ $\{\sin\frac{\pi}{6}, \sin\frac{2\pi}{6}, \sin\frac{3\pi}{6}, \ldots\}$

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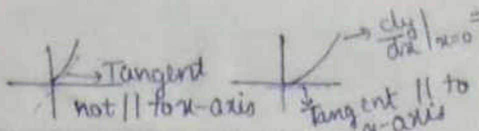


$\sin\frac{3\pi}{6} = \sin\frac{\pi}{2} = 1 \in A$ $\sin\frac{9\pi}{6} = \sin\frac{3\pi}{2} = -1 \in A$

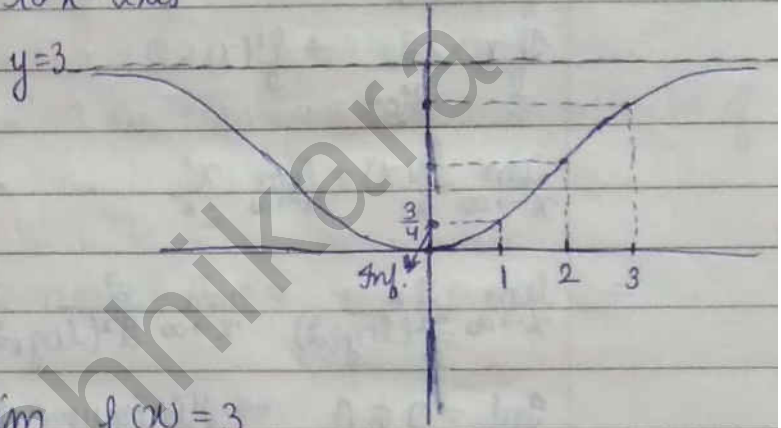
Sup & Maximum element Inf & Minimum element

(8) $\left\{\frac{3n^2}{n^2+3} : n \in \mathbb{N}\right\} \neq \left[\frac{3}{4}, 3\right) \rightarrow$ As it doesn't jump

$f(x) = \frac{3x^2}{x^2+3}$ is even

$f'(x) = \frac{(x^2+3)(6x) - 3x^2(2x)}{(x^2+3)^2} = \frac{18x}{(x^2+3)^2}$ >0 if $x>0$, <0 if $x<0$

$\frac{dy}{dx}\Big|_{x=0} = 0 \Rightarrow$ Tangent || to x-axis



$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x^2}{x^2+3}$

$= \lim_{x\to\infty} \frac{\cancel{6}^{3}x}{\cancel{2x}} = 3$

or $\lim_{x\to\infty} \frac{3}{1+\frac{3}{x^2}} = 3$

$\lim_{x\to-\infty} f(x) = 3$

$\Rightarrow \lim_{x\to\pm\infty} f(x) = 3$

$\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} \frac{18x}{(x^2+3)^2} = \lim_{x\to\infty} \frac{18}{2(x^2+3)\times 2x} = 0 \Rightarrow$ || to $y=3$

Sup $= 3 \notin A$ Inf $A = \frac{3}{4} \in A$

Maximum element doesn't exist $\rightarrow$ Minimum element

(9) $\left\{\frac{n^2}{2^n} : n \in \mathbb{N}\right\}$

$f(x) = \frac{x^2}{2^x}$

$f'(x) = \frac{2^x.2x - x^2.2^x.\log_e 2}{2^{2x}} = \frac{2x - x^2\log_e 2}{2^x} = \frac{x[2 - x\log_e 2]}{2^x}$

If $x>0$, then $2 - x\log_e 2 > 0 \Rightarrow (\log_e 2)x < 2$

$\log_e 2 > 0 \Rightarrow x < \frac{2}{\log_e 2} = \frac{2}{\log_{10} 2} \times \log_{10} e = \frac{2}{0.301} \times 0.4343 = 2.8$

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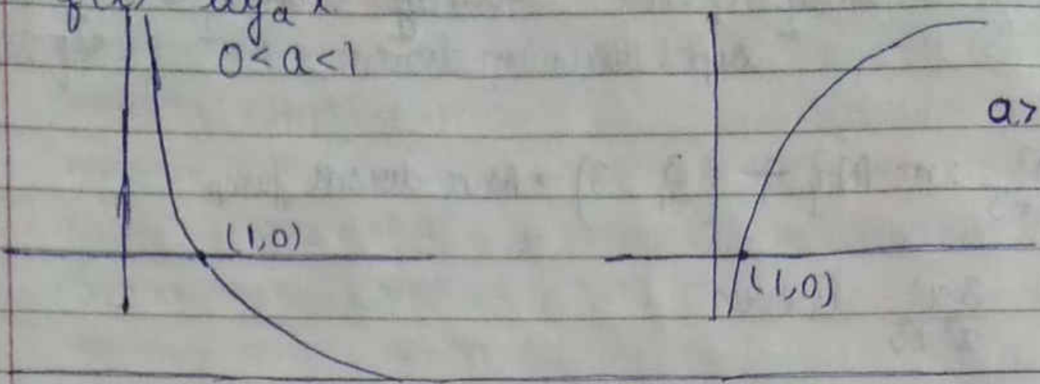
★ $f(x) = \log_a x$

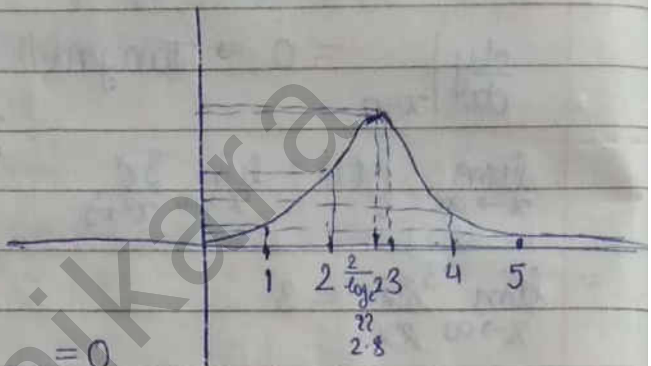
$0 < a < 1$



If $0 < x < \frac{2}{\log_e 2}$

$f'(x) > 0$

If $x > \frac{2}{\log_e 2} \Rightarrow f'(x) < 0$



$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^2}{2^x}$

$= \lim_{x \to \infty} \frac{2x}{2^x(\log_e 2)} = \lim_{x \to \infty} \frac{2}{2^x(\log_e 2)^2} = 0$

Inf. $= 0 \notin A \Rightarrow$ Minimum element doesn't exist

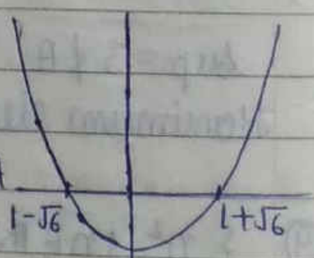
$n = 2 \Rightarrow \frac{n^2}{2^n} = \frac{2^2}{2^2} = 1$

$n = 3 \Rightarrow \frac{n^2}{2^n} = \frac{9}{8} > 1 = \text{Sup} \in A$ → Maximum element.

(10) $\{x \in \mathbb{R} : x^2 - 2x - 5 < 0\} = (1-\sqrt{6}, 1+\sqrt{6})$

$D = (\sqrt{4+20})^2 = 24 > 0$ Inf $\notin A$ Sup $\notin A$

$\Rightarrow x = \frac{2 \pm \sqrt{24}}{2} = 1 \pm \sqrt{6}$

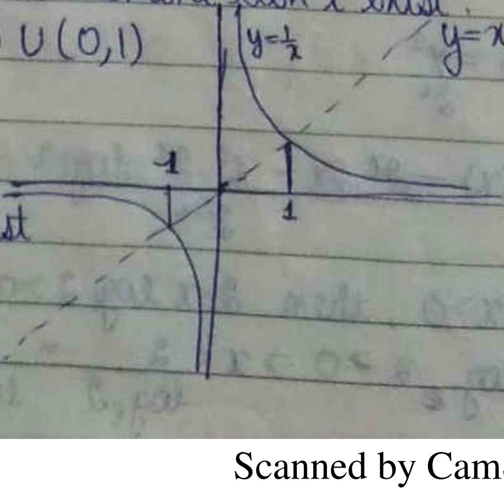


∴ Maximum and minimum element don't exist.

(11) $\{x \in \mathbb{R}^* : x < \frac{1}{x}\} = (-\infty, -1) \cup (0,1)$

Sup $= 1 \notin A$

Inf and Minimum and Maximum element don't exist



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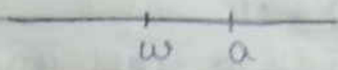
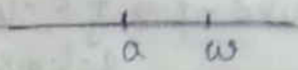
★ $A \neq \phi$, $A \subseteq \mathbb{R}$

If w is not an U.B. of A, then $\exists$ an element $a \in A$ s.t. $w < a$



If w is not a L.B. of A, then $\exists$ an element $a \in A$ s.t. $w > a$



★ $A \neq \phi$, $A \subseteq \mathbb{R}$

s: supremum of A

If $w < s$, then w is not an U.B. of A & $\exists$ an element $a_w \in A$ s.t. $w < a_w$ (depending on w)

★ Result: If $A \neq \phi$, $A \subseteq \mathbb{R}$, then a number s is a supremum of A iff.

① s: upper bound of A

② For any $\varepsilon > 0$, there exists an $a \in A$ s.t. $s - \varepsilon < a$

Q Which of the following is (are) true?

① inf. $\{n^{(-1)^n} : n \in \mathbb{N}\}$ doesn't exist.

✓② inf. $\{n^{(-1)^n} : n \in \mathbb{N}\}$ equals zero

✓③ $\{n^{(-1)^n} : n \in \mathbb{N}\}$ is unbounded above

✓④ $\{n^{(-1)^n} : n \in \mathbb{N}\}$ doesnot possess a smallest number.

Solⁿ: $\{\frac{1}{1}, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \dots\}$ → tend to 0   Inf. $= 0 \notin A$

Minimum element doesn't exist

As even nos. are infinite, ∴, sup not exists and hence maximum element doesn't exist.

Q Let A be a nonempty bounded below subset of $\mathbb{R}$. Prove $\inf A = -\sup\{-a : a \in A\}$

Solⁿ: $B = \{-a : a \in A\}$

Suppose $\inf A = t$

T.S: $\sup B = -t$

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① I.S. $-a \le -t \ \forall -a \in B$ i.e. $a \ge t \ \forall a \in A$ which is true
② $w < -t$
I.S. w is not an U.B. of B
$w < -t \Rightarrow t < -w \Rightarrow$ Not a L.B. of A $\Rightarrow \exists$ an $\alpha \in A$ s.t $\alpha < -w$
g.l.b. of A
$\Rightarrow$ $(-\alpha) > w \Rightarrow w$ is not an U.B. of B
element of B ($\because \alpha \in A$)

Q If A, B are non empty bounded above subsets of $\mathbb{R}$. Then show $\sup(A \cup B) = \sup\{\sup A, \sup B\}$

Sol^n: Let $\sup A = s_1$ $\sup B = s_2$
I.S. $\sup(A \cup B) = \sup\{s_1, s_2\} = s$ (i.e $s_1 \le s$ & $s_2 \le s$)
① $s \ge s_1 \ge a \ \forall a \in A$
$s \ge s_2 \ge b \ \forall b \in B$ $\Big] \Rightarrow s \ge c \ \forall c \in A \cup B$
② Let $w < s$
Case I $s = s_1$
$w < s$ i.e. $w < s_1 \Rightarrow w$ can't be a U.B. of A (as $\exists a \in A$ s.t. $w < a$) $\Rightarrow$ w can't be a U.B. of $A \cup B$
Case II: $s = s_2$
$w < s$ i.e. $w < s_2 \Rightarrow w$ can't be a U.B. of B (as $\exists b \in B$ s.t. $w < b$) $\Rightarrow$ w can't be a U.B. of $A \cup B$.
So, $\sup(A \cup B) = s$.

⊛ Ⓐ How to show t is an infimum of A?
① t: L.B. of A   ② If $w > t$, w can't be L.B.
Ⓑ How to show s is a supremum of A?
① s : U.B of A   ② If $w < s$, w can't be U.B.

⊛ A : non empty subset of $\mathbb{R}$
$\sup A = s$,
$t \in \mathbb{R}, t > s \Rightarrow t \notin A$

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If $X \subseteq Y$

then $\boxed{x \in X \Rightarrow x \in Y}$ is true

Q A : non empty bounded subset of $\mathbb{R}$

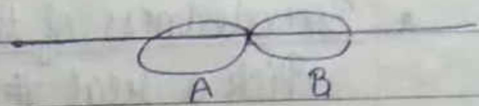
Show $\inf A \leq \sup. A$

Solⁿ Let $a \in A$

$\left.\begin{array}{l}\inf A \leq a \\ \sup A \geq a\end{array}\right] \Rightarrow \inf A \leq \sup A$

V.Imp

Q A, B : non empty subsets of $\mathbb{R}$



$a \leq b \;\forall a \in A, b \in B$

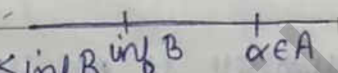
Show $\sup A \leq \inf B$.

Solⁿ Every element of B is an U.B. of A $\Rightarrow$ A is bounded above

It is sufficient to show inf B is an U.B. of A — ①

Deny ① There exists an element, say $\alpha \in A$ s.t. $\inf B < \alpha$

Case I: $\inf B \in B$?



$\inf B \in B$ & $\alpha \in A \Rightarrow \alpha \leq \inf B$

Also, by ~~number line~~ above statement $\inf B < \alpha \Rightarrow\Leftarrow (\because a \leq b)$

Case II: $\inf B \notin B$

Can α be a lower bound of B?

No, inf B is greatest l.b. of B

α : not a l.b. of B $\Rightarrow \exists$ a $\beta \in B$ s.t. $\beta < \alpha \Rightarrow\Leftarrow (\because a \leq b)$

Hence, our deny is wrong & ① is correct.

$\therefore \sup A \leq \inf B$ ($\because$ Sup A is the lowest U.B. of A)

OR Observe: Each element of B is an U.B. of A

$\therefore \sup A \leq b \;\forall b \in B$

$\Rightarrow$ Sup A is a lower bound of B.

$\Rightarrow \underset{\text{Greatest L.B. of B}}{\inf B} \geq \underset{\text{LB. of B}}{\sup A}$

★ Tautology (Logical Reasoning): $\boxed{P \Rightarrow Q}$ True | False

True: $P^{\checkmark} \Rightarrow Q^{\checkmark}$

$P^{\times}$

False: $P^{\checkmark} \Rightarrow Q^{\times}$

⊛ If P is not true then $\boxed{P \Rightarrow Q}$ is true.

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Is b an U.B. of X?
Check if $\boxed{x \in X \Rightarrow x \le b}$ is true.

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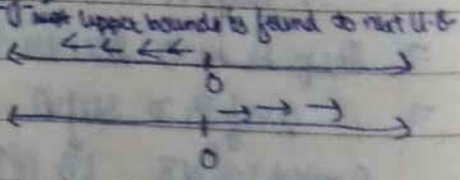
Q. Show that the empty set $\phi \subseteq A$, where A is any set.
Solⁿ: Check $\boxed{x \in \phi \Rightarrow x \in A}$ is true or Not? (or validity)
Yes it is true ∵ Never true

- Bounded set: Bounded above as well as bounded below

- Boundedness of the empty set ϕ:
  Pick a real # m
  Is 'm' an U.B. of ϕ?
  Check if it is true? Yes
  As $\boxed{\underbrace{x \in \phi}_{\text{Never true}} \Rightarrow x \le m}$ is true

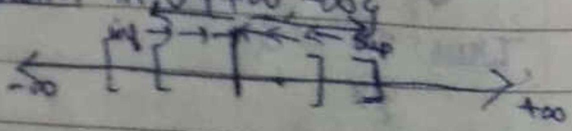
(*) Every real number is an upper bound of the empty set.
(*) Every real number is an lower bound of the empty set.
(*) The empty set is a bounded set.

- Axiom of Completeness of $\mathbb{R}$:
  If $\phi \ne A \subseteq \mathbb{R}$, A: bounded above, ~~then it~~ has the least upper bound
  Statement: Every non-empty subset of $\mathbb{R}$ that is bounded above, has the least upper bound.
  Significance of the word 'non-empty'?
  l.u.b. of ϕ → It doesn't exist
  g.l.b. of ϕ → It doesn't exist



(*) Supremum & Infimum of ϕ don't exist.

- Extended Real number system: $\mathbb{R} \cup \{+\infty, -\infty\}$
  $\left.\begin{array}{l} \text{Sup } \phi = -\infty \\ \text{Inf } \phi = +\infty \end{array}\right]$ strange but true.



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Trichotomy law
Tri → Three
chotus → Cutting

$\mathbb{R} = (-\infty, \infty)$
∞: lemniscate / lazy eight

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★ $A \neq \phi$, $A \subseteq \mathbb{R}$, A: Bounded
If $x \in A$, $\inf A \leq x < \sup A \Rightarrow \underline{\inf A \leq \sup A}$
Not true for empty set as $x \notin A$

★ When $\inf A = \sup A$?
When A is singleton, e.g., $A = \{1\}$

⊛ **Result**: The set of all rational numbers ($\mathbb{Q}$) is not complete.

Proof: Consider $A = \{a \in \mathbb{Q} : a \geq 0, a^2 < 2\}$
$1 \in A \Rightarrow A \neq \phi$
2 is an U.B of A ⇒ A is bounded above
Claim: A has no least U.B in $\mathbb{Q}$
Let if $\sup A = k$ — Trichotomy law

Case I: $k^2 = 2$
Impossible case!
($\because$ there is no such rational no. whose square is less than 2)

Case II: $k^2 < 2$
$\left.\begin{array}{l} y \in A \\ y > k \end{array}\right] \Rightarrow \Leftarrow$

Case III: $k^2 > 2$
$\left.\begin{array}{l} y \text{ is an U.B of A} \\ y < k \end{array}\right] \Rightarrow \Leftarrow$

$\therefore$ Our assumption is wrong ⇒ There doesn't exist any k.

Rough Work

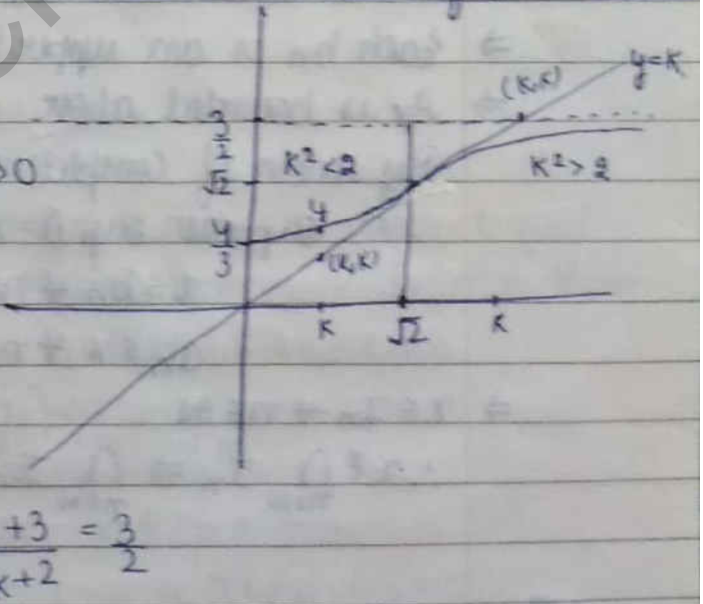
$$y = \frac{4+3K}{3+2K} \quad \left(y = \frac{2a+bK}{b+aK} \right)$$

$$\frac{dy}{dK} = \frac{3(3+2K) - 2(4+3K)}{(3+2K)^2} = \frac{1}{(3+2K)^2} > 0$$

$$y\Big|_{K=\sqrt{2}} = \frac{4+3\sqrt{2}}{3+2\sqrt{2}} = \frac{\sqrt{2}\cancel{(2\sqrt{2}+3)}}{\cancel{3+2\sqrt{2}}} = \sqrt{2}$$

Can't take. To make graph only

$$\frac{dy}{dK}\Big|_{K=0} = \frac{1}{9}$$



$$\lim_{K \to \infty} y = \lim_{K \to \infty} \frac{4+3K}{3+2K} = \lim_{K \to \infty} \frac{4/K + 3}{3/K + 2} = \frac{3}{2}$$

If $K \geq \in \mathbb{Q}$, then $y \in \mathbb{Q}$

Case II: if $k^2 < 2$

$$\left[\begin{array}{l} y \in A \to y^2 - 2 = \left(\left(\frac{4+3K}{3+2K}\right)^2 - 2\right) = \frac{K^2-2}{(3+2K)^2} < 0 \Rightarrow y^2 - 2 < 0 \Rightarrow y^2 < 2 \\ y > k \to y - K = \frac{4+3K}{3+2K} - K = \frac{4-2K^2}{3+2K} = \frac{2(2-k^2)}{3+2k} > 0 \Rightarrow y - k > 0 \text{ i.e. } y > k \end{array}\right.$$

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- Nested interval property:

$\longleftarrow \underset{a_1}{[}\ \underset{a_2}{[}\ \underset{a_3}{[} \longrightarrow \underset{b_3}{]}\ \underset{b_2}{]}\ \underset{b_1}{]} \longrightarrow$

$\bigcap_{n=1}^{\infty} I_n \neq \phi \Leftarrow \begin{cases} I_n = [a_n, b_n] \ \forall n \in \mathbb{N} \\ I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \xrightarrow{\text{non-empty}} \end{cases}$ → Nested sequence of closed intervals

★ $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4$

$A_1 \cap A_2 = A_2 \qquad A_1 \cap A_2 \cap A_3 = A_3 \qquad A_1 \cap A_2 \cap A_3 \cap A_4 = A_4$

- Doubt: $I_1 \cap I_2 \cap I_3 \cap \ldots \cap I_m \cap \ldots = \phi$?

We will prove $\bigcap_{n=1}^{\infty} I_n \neq \phi$

$A = \{a_1, a_2, a_3, \ldots\}$ → Left end pts. of intervals

$B = \{b_1, b_2, b_3, \ldots\}$ → Right end pts. of intervals

Claim: b_n is an upper bound of $A \ \forall n \in \mathbb{N}$

$m, n \in \mathbb{N}$

$m < n$	$m > n$	$m = n$
$a_m \le a_n \le b_n$	$a_m \le b_m \le b_n$	$a_m \le b_m = b_n$
$a_m \le b_n$	$a_m \le b_n$	$a_m \le b_n$

$\Rightarrow a_m \le b_n \ \forall m, n \in \mathbb{N}$

⇒ Each b_n is an upper bound of A.

⇒ A is bounded above

By Axiom of completeness, sup A exists.

Suppose sup $A = x$

$\left. \begin{array}{l} x \le b_n \ \forall n \in \mathbb{N} \\ a_n \le x \ \forall n \in \mathbb{N} \end{array} \right\} \Rightarrow a_n \le x \le b_n \ \forall n \in \mathbb{N}$

$\Rightarrow x \in I_n \ \forall n \in \mathbb{N}$

$\Rightarrow x \in \bigcap_{n \in \mathbb{N}} I_n \Rightarrow \bigcap_{n \in \mathbb{N}} I_n \neq \phi$

imp.

- Archimedean property:

(i) Given any real number x, there exists a natural number n s.t. $x < n$

Proof: Let if possible, $\mathbb{N}$ be bounded above.

AoC: sup $\mathbb{N}$ exists

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$\sup \mathbb{N} = \alpha$

$n \le \alpha \ \forall n \in \mathbb{N}$

$\alpha - 1$: Is it an U.B. of $\mathbb{N}$? No

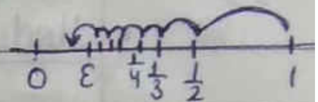
$\exists$ an $m \in \mathbb{N}$ s.t. $\alpha - 1 < m \Rightarrow \alpha < m+1 \Rightarrow \Leftarrow$ ($\because m+1 \in \mathbb{N}$ as $m \in \mathbb{N}$)

$\therefore$ Our assumption is wrong.

(ii) Given any positive real number ε, $\exists$ a natural number n s.t. $\frac{1}{n} < \varepsilon$

Proof: Let $\varepsilon > 0$ be any ~~+ve~~ real no, $\therefore \frac{1}{\varepsilon} > 0$ is also a real no.

By (i), $\exists$ a natural no. n s.t. $\frac{1}{\varepsilon} < n \Rightarrow \frac{1}{n} < \varepsilon$



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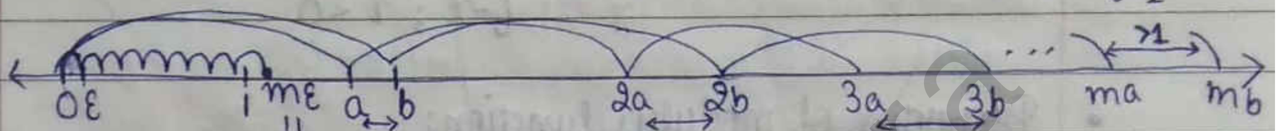


$m(b-a) = mb - ma$.

$\varepsilon = b - a$

$m\varepsilon > 1 \Rightarrow mb - ma > 1$

$a < b \Rightarrow b - a > 0 \Rightarrow \varepsilon > 0 \Rightarrow$

$\exists$ a natural no. m s.t. $\frac{1}{m} < \varepsilon$

i.e. $m\varepsilon > 1$

★ There exists a natural number, say n, s.t. $ma < n < mb$

$\Rightarrow a < \frac{n}{m} < b$

$\frac{n}{m} \to$ Rational number

⊛ Result (Density theorem for $\mathbb{Q}$): Given any two distinct real numbers $a < b$, there exists a rational number between them.

$0 < a < b \Rightarrow 0 < ma < mb \Rightarrow \exists$ a $n \in \mathbb{N}$ s.t. $ma < n < mb$

$\Rightarrow 0 < ma < n < mb \Rightarrow 0 < a < \frac{n}{m} < b$

$\frac{n}{m} \to$ Rational no.

$a < b < 0 \Rightarrow ma < mb < 0$

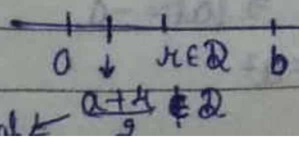
$\Rightarrow 0 < -mb < -ma \Rightarrow \exists$ a $n \in \mathbb{N}$ s.t. $-mb < n < -ma$

$\Rightarrow 0 < -mb < n < -ma \Rightarrow 0 < -b < \frac{n}{m} < -a \Rightarrow a < -\frac{n}{m} < b < 0$

$-\frac{n}{m} \to$ Rational no.

⊛ Result (Density theorem for $\mathbb{R} \setminus \mathbb{Q}$): Given any two distinct real numbers, there exists an irrational number between them.

Case I: $a \notin \mathbb{Q}$



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Case 2: $a \in \mathbb{Q}$ $\qquad \left(\frac{a+r}{2} \in \mathbb{Q}\right)$

$$\overset{b-a}{\longleftrightarrow}$$
$a \quad a+\frac{\sqrt{2}}{n} \quad r \quad b$
$a+\frac{\sqrt{2}}{n} \to$ Irrational

Is $\frac{\sqrt{2}}{n} < b-a$? Yes

Let $\frac{b-a}{\sqrt{2}} = \varepsilon$

We get an $n \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon \Rightarrow \frac{\sqrt{2}}{n} < b-a$.

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- Modulus function: $|x| = \begin{cases} x \ ; \ x > 0 \\ 0 \ ; \ x = 0 \\ -x \ ; \ x < 0 \end{cases}$

- Properties of modulus function:

(1) $a, b \in \mathbb{R}$

$|ab| = |a||b|$

Case I: $a \geq 0, b \geq 0 \Rightarrow |a| = a, |b| = b$

$ab \geq 0 \Rightarrow |ab| = ab$

$|a||b| = ab = |ab|$

Case II: $a < 0, b \geq 0 \Rightarrow |a| = -a, |b| = b$

$ab \leq 0 \Rightarrow |ab| = -ab$

$|a||b| = -ab = |ab|$

Case III: $a \geq 0, b \leq 0 \Rightarrow |a| = a, |b| = -b$

$ab \leq 0 \Rightarrow |ab| = -ab$

$|a||b| = +a.(-b) = -ab = |ab|$

Case IV: $a \leq 0, b \leq 0 \Rightarrow |a| = -a, |b| = -b$

$ab \geq 0 \Rightarrow |ab| = +ab$

$|a||b| = (-a).(-b) = ab = |ab|$

(2) $|x| = \max.\{x, -x\} \Rightarrow x \leq |x|$

(3) $|a|^2 = a^2$

Case I: $a \leq 0 \Rightarrow |a| = -a$

$|a|^2 = (-a)^2 = a^2$

Case II: $a \geq 0 \Rightarrow |a| = a$

$|a|^2 = a^2$

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$x^2<y^2 \not\Rightarrow x<y$

e.g. $(2)^2<(-3)^2 \not\Rightarrow 2<-3$

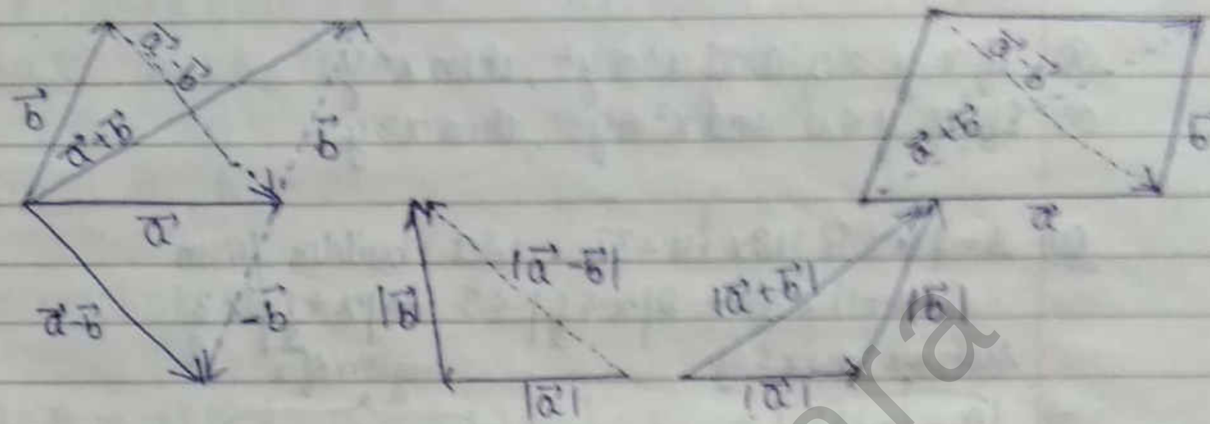
④ $c \geq 0$, $|a|$: distance of 'a' from '0'

$|a| \leq c \Leftrightarrow -c \leq a \leq c$

⑤ $c>0$

$|a|<c \Leftrightarrow -c<a<c$

★



⊛ Sum of any two sides of a triangle exceeds the third side.

⊛ Difference of any two sides of a triangle is less than the third side.

⊛ ① $|\vec{a}|+|\vec{b}| \geq |\vec{a}+\vec{b}|$

② $|\vec{a}|+|\vec{b}| \geq |\vec{a}-\vec{b}|$

③ $\big||\vec{a}|-|\vec{b}|\big| \leq |\vec{a}-\vec{b}|$

$\xrightarrow{\vec{a}+\vec{b}}$ $\vec{a}$ $\vec{b}$ parallel (not form Δ)

⊛ $a, b \in \mathbb{R}$

① $|a+b| \leq |a|+|b|$

② $|a-b| \leq |a|+|b|$

③ $\big||a|-|b|\big| \leq |a-b|$

} Triangle inequalities (as we proved with help of triangle)

Proof: ① $(L.H.S)^2 = |a+b|^2 = (a+b)^2 = a^2+b^2+2ab$ — ①

$(R.H.S)^2 = (|a|+|b|)^2 = |a|^2+|b|^2+2|a||b| = a^2+b^2+2|ab|$ — ②

Compare ① & ②

As $ab \leq |ab| \Rightarrow (LHS)^2 \leq (RHS)^2 \Rightarrow LHS \leq RHS$ ($\because LHS, RHS \geq 0$)

When L.H.S = R.H.S? when $ab \geq 0$

② Put $b=-b$ in ① ($-b$ at place of b)

$|a+(-b)| = ~~|a-b|~~ = |a|+|-b| \leq |a|+|b|$

i.e. $|a-b| = |a|+|b|$

③ Put $a=a-b$ in ①

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$||(a-b)+b| \leq |a-b|+|b| \Rightarrow |a|-|b| \leq |a-b|$ — Ⓐ

Put $b = b-a$ in ①

$|a+(b-a)| \leq |a| + |b-a| \Rightarrow -|a|+|b| \leq |b-a|$

$\Rightarrow -(|a|-|b|) \leq |a-b|$ — Ⓑ

Ⓐ & Ⓑ $\Rightarrow ||a|-|b|| \leq |a-b|$

⊛ If $x, y \geq 0$ and $x^2 \leq y^2$, then $x \leq y$

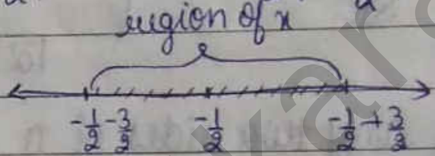
⊛ If $x, y \leq 0$ and $x^2 \leq y^2$, then $x \geq y$

Q① $A = \{x \in \mathbb{R} : |2x+1| < 3\}$ → Set-builder form

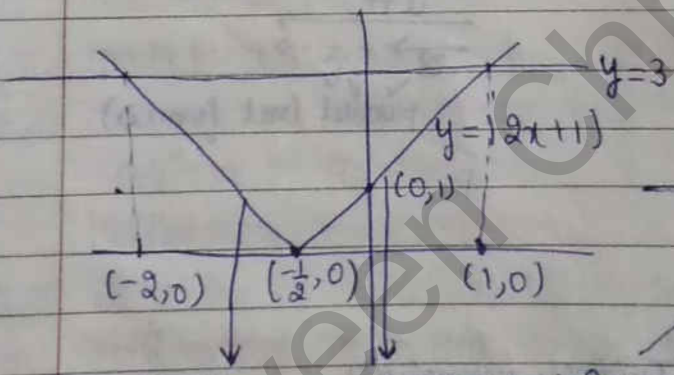
$|2x+1| < 3 \Rightarrow 2\left|x+\frac{1}{2}\right| < 3 \Rightarrow \left|x+\frac{1}{2}\right| < \frac{3}{2}$

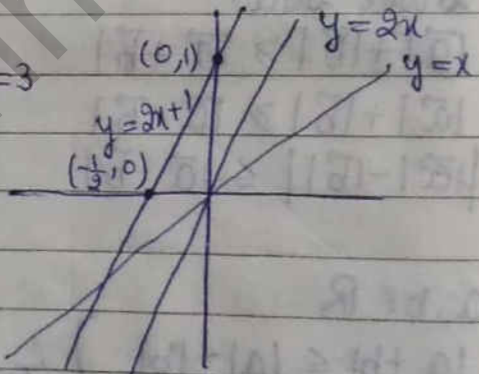
distance b/w x & $-\frac{1}{2}$

$\Rightarrow \left|x-\left(-\frac{1}{2}\right)\right| < \frac{3}{2}$



$x \in \left(-\frac{1}{2}-\frac{3}{2}, -\frac{1}{2}+\frac{3}{2}\right)$ i.e. $x \in (-2, 1)$

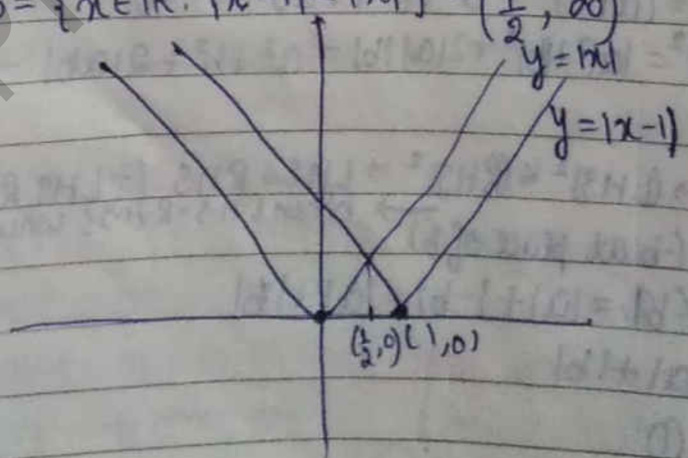




$y = -(2x+1)$ $\quad$ $y = 2x+1 \Rightarrow 2x+1 = 3 \Rightarrow x = 1$

$\Rightarrow -(2x+1) = 3 \Rightarrow x = -2$

② $B = \{x \in \mathbb{R} : |x-1| < |x|\} = \left(\frac{1}{2}, \infty\right)$



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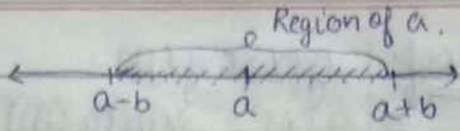
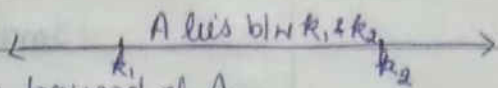
⊛ $|x-a|<b \Rightarrow b>0 \; (\because |x-a| \geq 0)$



$\Rightarrow x \in (a-b,\ a+b)$

$|x-a| \leq b \Rightarrow b \geq 0 \Rightarrow x \in [a-b, a+b]$

★ A: Bounded non empty subset of $\mathbb{R}$.



k_1: lower bound of A $\quad$ k_2: upper bound of A

Case I: $k_1, k_2 > 0$

Case II: $k_1, k_2 < 0$

Case III: $k_1 < 0, k_2 > 0$

$x \in A \Rightarrow |x| \leq \{\max.\{|k_1|, |k_2|\}\} = K$

⊛ If A is a non empty bounded subset of $\mathbb{R}$, then $\exists$ a $K>0$ s.t.

$\underline{|x| \leq K} \ \forall x \in A$

$-K \leq x \leq K$

★ $f: A \to \mathbb{R}$

① If Range of f is bounded above, then f is called a bounded above function.

② If Range of f is bounded below, then f is called a bounded below function.

③ If Range of f is bounded, then f is called a bounded function.

E.g. $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$

Range$(f) = [0, \infty)$

f is bounded below but not bounded above.

★ $f, g: A \to \mathbb{R}$, $A \neq \phi$ → Bounded func^n.

$f(x) \leq g(x) \ \forall x \in A$

$f(A)$: range of $f = \{f(x) : x \in A\}$ $\quad$ $g(A)$: range of $g = \{g(x) : x \in A\}$

then show $\sup(f(A)) \leq \sup(g(A))$

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	f	g	
	5N	7N	
Sup = 6 →	3N	4N	→ Inf 4
	6N	10N	

Proof: $g(x) \in g(A)$

$\Rightarrow g(x) \le \sup(g(A)) \ \forall x \in A$ }
$f(x) \le g(x) \ \forall x \in A$ } $\Rightarrow f(x) \le \sup(g(A)) \ \forall x \in A$

$\Downarrow$

$\sup(g(A))$ is an U.B. of $f(A)$

$\Downarrow$

Smallest U.B. $\leftarrow \sup(f(A)) \le \sup(g(A)) \rightarrow$ Any U.B.

JAM 2015

Let $f, g : [0,1] \to [0,1]$ be bounded functions. Suppose $R(f)$ & $R(g)$ represent their respective ranges. Which of the following is (are) true?

(a) $f(x) \le g(x)$ for all $x \in [0,1] \Rightarrow \sup R(f) \le \inf R(g)$

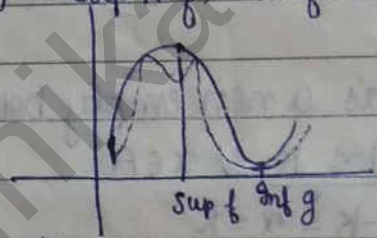
✓(b) $f(x) \le g(x)$ for some $x \in [0,1] \Rightarrow \inf R(f) \le \sup R(g)$

✓(c) $f(x) \le g(y)$ for some $x, y \in [0,1] \Rightarrow \inf R(f) \le \sup R(g)$

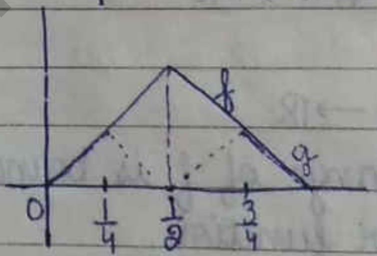
✓(d) $f(x) \le g(y)$ for all $x, y \in [0,1] \Rightarrow \sup R(f) \le \inf R(g)$

Sol^n: (a) $\sup R(f) \not\le \inf R(g)$

e.g. $g(x) = \begin{cases} x & ; 0 \le x \le \frac{1}{2} \\ 1-x & \frac{1}{2} \le x \le 1 \end{cases}$



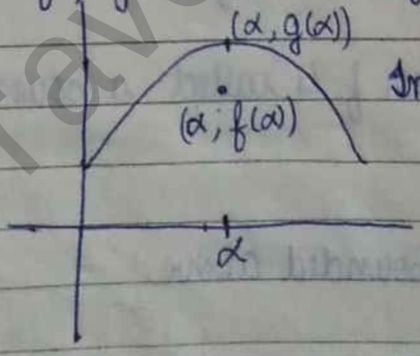
$f(x) = \begin{cases} x & ; 0 \le x \le \frac{1}{4} \\ \frac{1}{2} - x & ; \frac{1}{4} \le x \le \frac{1}{2} \\ x - \frac{1}{2} & ; \frac{1}{2} \le x \le \frac{3}{4} \\ 1 - x & ; \frac{3}{4} \le x \le 1 \end{cases}$



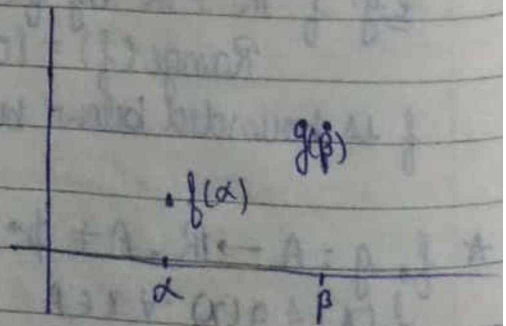
Or

Let $f = g$, then $\sup R(f) \ge \inf R(f) = \inf R(g)$

(b)



$\inf R(f) \le f(\alpha) \le g(\alpha) \le \sup R(g)$



(c) $\inf. R(f) \le f(\alpha) \le g(\beta) \le \sup R(g)$

(d) $\left[\begin{array}{l} a \le b \ \forall a \in A, \forall b \in B \\ \Rightarrow \sup A \le \inf B \end{array} \right]$ (Pg-25)

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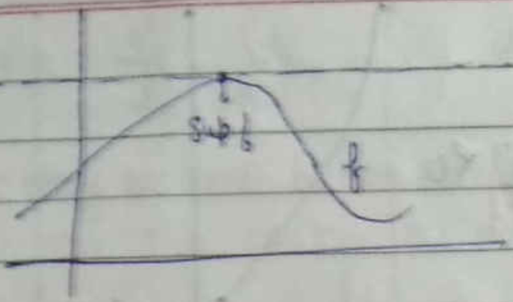
One-to-one correspondence → Bijective func^n
↕
one-one & onto func^n

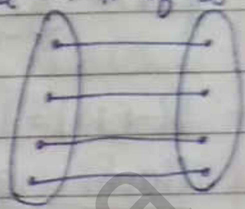


g [can't be below this line as $a \le b \ \forall a \in R(f) \ \forall b \in R(g)$]

$\Rightarrow \sup R(f) \le \inf R(g)$

- Countability of sets:

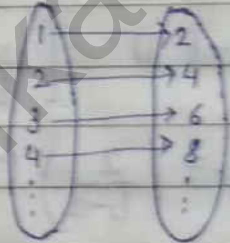
[Rational & irrationals might be in the same proportion in (a, b)] → Wrong → proved by George Cantor
$\#$ of $\mathbb{R} \setminus \mathbb{Q} >> \#$ of $\mathbb{Q}$



A, B: sets
Cardinality: size of set
A & B have the same cardinality if ∃ exists a one-to-one & onto func^n b/w A & B

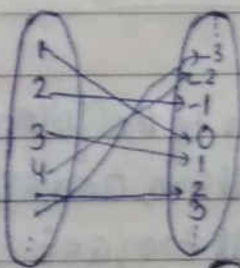
★ $\mathbb{N}, E$

$f: \mathbb{N} \to E$ by $f(n) = 2n$ → Bijective



$E \subsetneq \mathbb{N}$
same cardinality
$|E| = |\mathbb{N}|$ → only when cardinality is infinite
Despite $E \subsetneq \mathbb{N}$, E & N have the same cardinality.

★ $\mathbb{N}, \mathbb{Z}$



n: even ; $f(n) = -\frac{n}{2}$
n: odd ; $f(n) = \frac{n-1}{2}$
$f: \mathbb{N} \to \mathbb{Z}$

Bijective ← $f(n) = \begin{cases} -\frac{n}{2} ; & n: \text{even} \\ \frac{n-1}{2} ; & n: \text{odd} \end{cases}$

$|\mathbb{N}| = |\mathbb{Z}|$

★ $n = 3m$ $\qquad$ $n = 3m-2$ $\qquad$ $n = 3m+2$

$f(n) = \frac{n}{3}$ $\qquad$ $f(n) = \left(\frac{n+2}{3}\right)i$ $\qquad$ $f(n) = -\left(\frac{n+1}{3}\right)$

$1 \to 1 \cdot i$ $\quad$ $4 \to 2 \cdot i$ $\quad$ $7 \to 3 \cdot i$

$2 \to -1$ $\quad$ $5 \to -2$ $\quad$ $8 \to -3$

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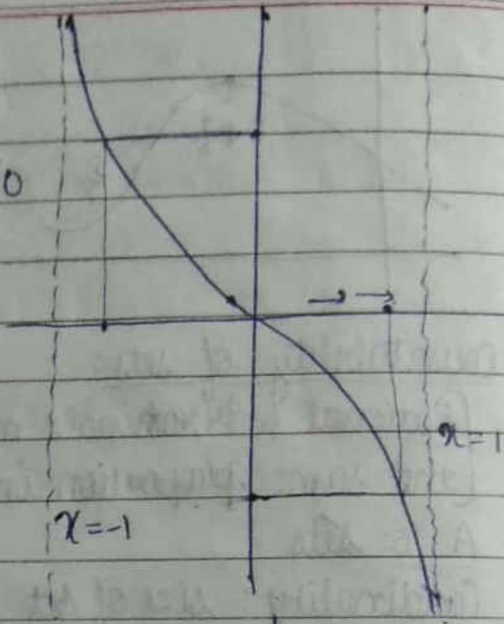
Assymptote → when deno. becomes zero

★ $y = \frac{x}{x^2-1} = \frac{x}{(x+1)(x-1)}$ → odd

$\frac{dy}{dx} = \frac{x^2-1-2x^2}{(x^2-1)^2} = -\left[\frac{x^2+1}{(x^2-1)^2}\right] < 0$

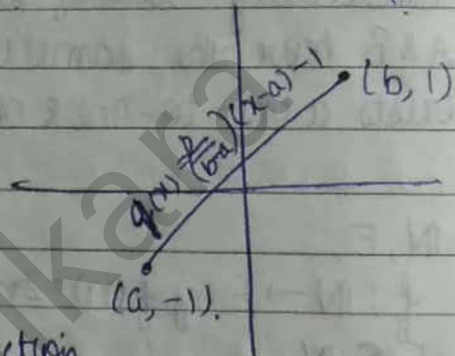
$\left.\frac{dy}{dx}\right|_0 = -1$



$\left.\begin{array}{l} f:(-1,1)\to\mathbb{R} \\ f(x) = \frac{x}{x^2-1}\end{array}\right\}$ → Bijection

⇒ $|(-1,1)| = |\mathbb{R}|$

Claim: $|(a,b)| = |\mathbb{R}|$, $a<b$.



Proof: $g:(a,b)\to(-1,1)$

$y-(-1) = \left[\frac{1-(-1)}{b-a}\right](x-a)$

⇒ $g = y = \left(\frac{2}{b-a}\right)(x-a)-1$ → Bijection

$f\circ g : (a,b)\to\mathbb{R}$ ⇒ $|(a,b)| = |\mathbb{R}|$
Bijective (as f & g are bijective)

20/7/16

DU 2014 Which of the following doesn't imply that $a=0$?

① For all $\varepsilon>0$, $0\le a<\varepsilon$ ② For all $\varepsilon>0$, $-\varepsilon\le a<\varepsilon$

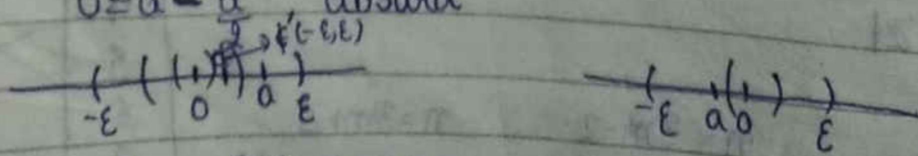
③ For all $\varepsilon>0$, $a<\varepsilon$ ④ For all $\varepsilon>0$, $0\le a\le\varepsilon$

Solⁿ: ① Let if possible, $a\ne 0$; Then $a>0$

Set $\varepsilon = \frac{a}{2}$

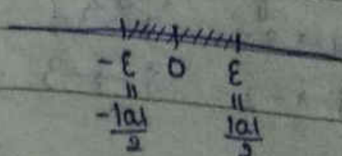
$0\le a<\frac{a}{2}$, absurd

② $a \notin (-\varepsilon,\varepsilon)$



Let if possible, $a\ne 0$

Set $\varepsilon = \frac{|a|}{2}$



$a \notin \left(\underset{\uparrow -\varepsilon}{-\frac{|a|}{2}}, \underset{\uparrow \varepsilon}{\frac{|a|}{2}}\right)$

③ a may be zero but a may be negative also ⇒ $a \neq 0$
a can't be positive as $a < \varepsilon$.

④ $0 \le a \le \frac{a}{2}$. absurd.

DU 2015 Which of the following are true?

(a) $\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1\right] = [0,1] \rightarrow (0,1]$ ✓(b) $\bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 1\right] = (0,1]$

(c) $\bigcap_{n=1}^{\infty} \left(1-\frac{1}{n}, 2\right] = (1,2) \rightarrow [1,2]$ ✓(d) $\bigcap_{n=1}^{\infty} \left[1-\frac{1}{n}, 2\right] = [1,2]$

Solⁿ: (a) $0 \quad \varepsilon \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{1}{2} \quad 1$ (large n) Whatever be $\varepsilon > 0$, however small ∃ an n_0 (fixed natural no.) $\in \mathbb{N}$ s.t. $\frac{1}{n_0} < \varepsilon$

$[0,1]$ ✗ $[0.0000001, 1]$ ✗

(b) $0 \; \cdots \; \frac{1}{4} \; \frac{1}{3} \; \frac{1}{2} \; 1$

(c) $0 \; \frac{1}{2} \; \frac{2}{3} \; \frac{3}{4} \; \frac{4}{5} \; 1\text{-}\varepsilon \; 1 \; 2$ $n\uparrow \Rightarrow \frac{1}{n}\downarrow \Rightarrow 1-\frac{1}{n} \longrightarrow 1$

$\lim_{n\to\infty} 1-\frac{1}{n} = 1$ & $1-\varepsilon \notin \left(1-\frac{1}{n}, 2\right]$

$1 \in \left(\underbrace{1-\frac{1}{n}}_{<1}, 2\right] \; \forall n \in \mathbb{N} \Rightarrow 1 \in \bigcap_{n=1}^{\infty} \left(1-\frac{1}{n}, 2\right]$

$0 \quad 1\text{-}\varepsilon \quad 1-\frac{1}{n_0} \quad 1 \quad 2$

For $\varepsilon > 0$, ∃ an $n_0 \in \mathbb{N}$ s.t. $\frac{1}{n_0} < \varepsilon \Rightarrow -\frac{1}{n_0} > -\varepsilon \Rightarrow 1-\frac{1}{n_0} > 1-\varepsilon$
$1-\varepsilon \in \left(1-\frac{1}{n_0}, 2\right]$? No $\left(\because 1-\varepsilon < 1-\frac{1}{n_0}\right)$

$\therefore 1-\varepsilon \notin \left(1-\frac{1}{n_0}, 2\right] \Rightarrow 1-\varepsilon \notin \bigcap_{n=1}^{\infty} \left(1-\frac{1}{n}, 2\right]$

No number less than 1 is in the intersection
Is 1 in the intersection? Yes ($\because$ 1 is in each set)

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$A \subseteq B,\ A \cup B = B$ $\quad$ $A \supseteq B,\ A \cap B = B$

$A \subseteq B \subseteq C,\ A \cup B \cup C = C$ $\quad$ $A \supseteq B \supseteq C,\ A \cap B \cap C = C$

Q ① $\bigcap_{n=1}^{\infty} \left[\underbrace{3-\frac{1}{n}}_{<3}, \underbrace{5+\frac{1}{n}}_{>5}\right]$ ② $\bigcup_{n=1}^{\infty} \left(2+\frac{1}{n}, 5-\frac{1}{n}\right)$

③ $\bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right)$ → Nested but intervals are not closed.

Sol^n: ① $[3,5]$

② $(2,5)$

$\bigcup_{n=1}^{\infty} \left[2+\frac{1}{n}, 5-\frac{1}{n}\right] = (2,5)$

③ $\bigcap_{n=1}^{\infty} \left[0, \frac{1}{n}\right) = \{0\}$

$\bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right) = \phi \ (\because 0 \notin (0, \frac{1}{n}))$

Nested but not closed intervals

If $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$ closed intervals, then $\bigcap_{n=1}^{\infty} I_n \neq \phi$

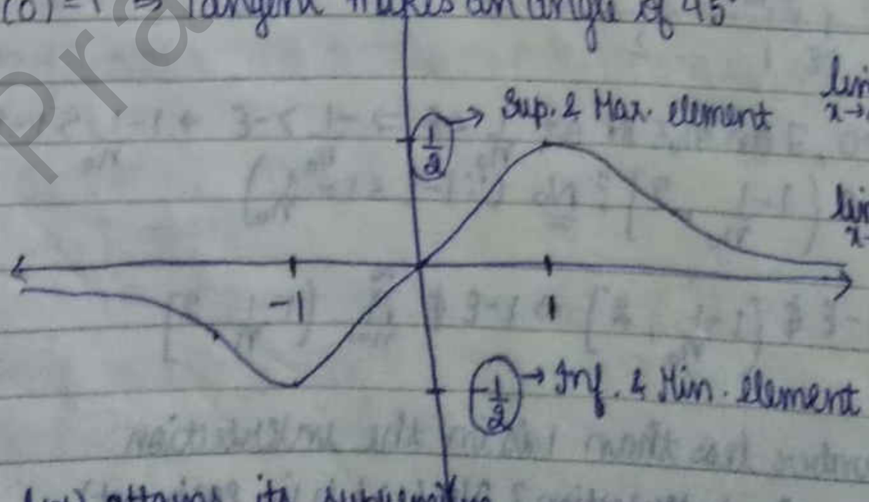
It is necessary to mention "closed" in Nested Interval Property

CMI 2011 The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{x}{x^2+1}$ (odd) attains its supremum.

Sol^n: $y = \frac{x}{x^2+1} \Rightarrow \frac{dy}{dx} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = \frac{(1-x)(1+x)}{(x^2+1)^2}$

$f(x)|_{x=0} = 0$ $\quad$ slope $-$ ... -1 ... $+$ (graph moves upward) ... 0 ... 1 ... $-$ (graph moves downward)

$f'(0) = 1 \Rightarrow$ Tangent makes an angle of $45°$



$\lim_{x\to\infty} f(x) = \frac{1/x}{1+\frac{1}{x^2}} = 0$

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x^2+1}$

$= \lim_{x\to\infty} \frac{1}{2x} = 0$

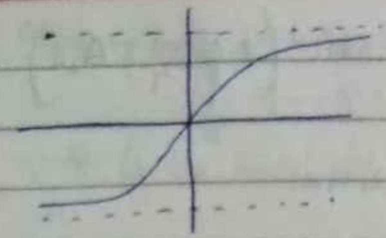
Yes, $f(x)$ attains its supremum

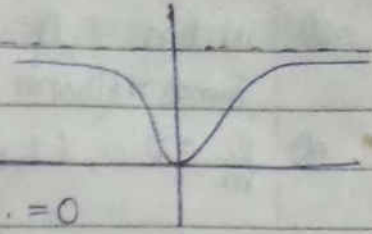
g.c.d(0,0) → Not defined g.c.d.(n,0) = |n|

[p,q] → L.C.M.

(p,q) → ~~H~~G.C.D.

★





Attain Inf. = 0

Neither attains sup. nor inf. Here, f doesn't attain its sup.

★ "A ~ B" means A & B have the same cardinality.

- Countable sets: A: any set
  If $\mathbb{N} \sim A$, then A is s.t.b a countable set.

- Uncountable sets: Infinite set & Not countable

⊛ E, $\mathbb{Z}$ are countable set ($\because \mathbb{N} \sim E$ & $\mathbb{N} \sim \mathbb{Z}$)

- A: countable set $\Rightarrow \mathbb{N} \sim A$

$\Rightarrow$ There exists a one-to-one correspondence $f: \mathbb{N} \to A$

Range of f ← $A = \{f(1), f(2), f(3), \ldots\}$ → Enumeration of A

⊛ A countable set can be enumerated as $\{a_1, a_2, a_3, \ldots\}$

⊛ Result: The set $\mathbb{Q}$ of rationals is countable

$$A_n = \{\pm \frac{p}{q} : p, q \in \mathbb{N}, q \neq 0, (p,q) = 1, p+q = n\}, \quad n \geq 2$$

$A_1 = \{\frac{0}{1}\}$ → Special case, we consider $n = 0 \notin \mathbb{N}$ in this case only

$A_2 = \{\frac{1}{1}, \frac{-1}{1}, \frac{2}{0}\}$ $A_3 = \{\frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1}\}$

$\frac{2}{0}$ ↳ is rejected g.c.d(2,0) = 2

$A_4 = \{\frac{1}{3}, \frac{-1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{-3}{1}\}$ $A_5 = \{\frac{1}{4}, \frac{-1}{4}, \frac{2}{3}, \frac{-2}{3}, \frac{3}{2}, \frac{-3}{2}, \frac{4}{1}, \frac{-4}{1}\}$

$\frac{2}{2}$ ↳ is rejected g.c.d.(2,2) = 2, $\frac{2}{2} = \frac{1}{1}$ & 1+1 = 2

⊛ $|A_n| < 2({}^{n+x-1}C_{x-1})$, $|A_n|$ is finite & is less than ~~equals to~~ $2({}^{n+2-1}C_{2-1})$

⊛ A_n is finite $\forall n \in \mathbb{N} \Rightarrow$ Onto

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$n=30$: identical mangoes
$r=5$: person
Ways of distributing : ${}^{n+r-1}C_{r-1}$

⊛ $m \neq n \Rightarrow \underset{\substack{\downarrow \\ \text{Sum of N\&D is } m}}{A_m} \cap \underset{\substack{\downarrow \\ \text{Sum of N\&D is } n}}{A_n} = \phi \Rightarrow$ One-to-One $\left(\text{e.g. } \frac{22}{7} \in A_{29}\right)$

⊛ $\frac{l}{m} \in \mathbb{Q}$, $(l,m)=1$, then $\frac{l}{m} \in A_{|l|+|m|}$

⊛ Every rational no. appears in exactly one A_n.

$$\begin{array}{lccccccccccc} \mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \mathbb{Q}: & \underbrace{\tfrac{0}{1}}_{A_1} & \underbrace{\tfrac{1}{1} \quad \tfrac{-1}{1}}_{A_2} & & \underbrace{\tfrac{1}{2} \quad \tfrac{-1}{2} \quad \tfrac{2}{1} \quad \tfrac{-2}{1}}_{A_3} & & & & \underbrace{\tfrac{1}{3} \quad \tfrac{-1}{3} \quad \cdots}_{A_4} & & \end{array}$$

One-to-one & onto

$\mathbb{N} \sim \mathbb{Q} \Rightarrow \mathbb{Q}$ is countable
So, $\mathbb{Q}$ is enumerable.

⊛ <u>Result</u>: The set $\mathbb{R}$ of all real numbers is uncountable.

Proof: Let if possible, $\mathbb{R}$ be countable
$\therefore \mathbb{R}$ can be enumerated as $\mathbb{R} = \{a_1, a_2, a_3, \ldots\} \rightarrow$ list
We are sure that each real number appears in this list.
By N.I.P., we will show that there is a real no. which is <u>NOT</u> in the list.

$[\ [\ [\ [\ \ ]_{I_4}\ \overset{a_4}{\cdot}\ ]_{I_3}\ ]^{I_2}\ \overset{\cdot}{a_2}\ \overset{\cdot}{a_3}\ ]^{I_1}\ \overset{\cdot}{a_1}$

Take any non-empty closed interval I_1 s.t. $a_1 \notin I_1$
Take any non-empty closed interval $I_2 \subseteq I_1$ s.t. $a_2 \notin I_2$.
Take any non-empty closed interval $I_3 \subseteq I_2$ s.t. $a_3 \notin I_3$
⋮
Take any non empty closed interval $I_{n+1} \subseteq I_n$ s.t. $a_{n+1} \notin I_{n+1}$
So, we get a nest of closed intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$ s.t. $a_{n+1} \notin I_{n+1}$
Pick a number, say a_{n_0} (fixed) from the list,
We have $a_{n_0} \notin I_{n_0}$
$\Rightarrow a_{n_0} \notin \bigcap_{n=1}^{\infty} I_n \Rightarrow \bigcap_{n=1}^{\infty} I_n = \phi$ — ①

Using, NIP, $\bigcap_{n=1}^{\infty} I_n \neq \phi \Rightarrow \exists$ a ~~real no.~~ $\underset{\downarrow}{x} \in \mathbb{R}$ s.t. $x \in \bigcap_{n=1}^{\infty} I_n$ — ②
Range $\neq$ Codomain $\Leftarrow$ Not in Range $\Leftarrow$ Outside the list
① & ② are contradictory $\Rightarrow \nexists$ func^n $f: \mathbb{N} \rightarrow \mathbb{R}$ which is onto

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(One-to-one doesn't create problem, only onto creates problem
$f: \mathbb{N} \to \mathbb{R}$ s.t. $f(n) = n \to$ One-to-one)

$\therefore \mathbb{R}$ is uncountable.

★ A, B: countable sets

$A \cup B$: Is it countable?

$a_1 \; b_1$ W.L.O.G, we can assume $A \cap B = \phi$

$a_2 \; b_2$ as if $A \cap B \neq \phi$, then $f: \mathbb{N} \to A \cup B$ is not one-to-one

$a_3 \; b_3$

$\vdots \quad \vdots$

$A \to f: \mathbb{N} \to A \to$ Bijection

$B \to g: \mathbb{N} \to B$

Bijection $h: \mathbb{N} \to A \cup B$ by $h(n) = \begin{cases} f\left(\frac{n+1}{2}\right) & ; n: \text{odd} \\ g\left(\frac{n}{2}\right) & ; n: \text{even} \end{cases}$

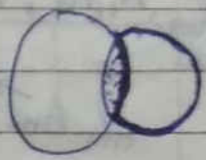
$h(1) = f(1) \quad h(2) = g(1) \quad h(3) = f(2) \quad h(4) = g(2) \; \cdots$

or $h(n) = \begin{cases} a_{\frac{n+1}{2}} & ; n: \text{odd} \\ b_{\frac{n}{2}} & ; n: \text{even} \end{cases}$ $\quad h(1) = a_1, \; h(2) = b_1, \; h(3) = a_2, \; h(4) = b_2, \; \vdots$

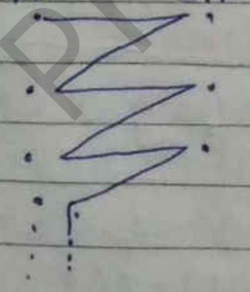
If $A \cap B \neq \phi$, then we replace B by $B|A$

$A \cup B = A \cup (B|A)$; $B|A \subseteq B \to$ countable

If $B|A$ is finite, then $\quad$ finite or countable



A $\quad$ B|A



⊛ The union of 2 countable sets is countable.

• <u>Cardinality of $\mathbb{R} \setminus \mathbb{Q}$?</u>

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Let if possible, $\mathbb{R}\setminus\mathbb{Q}$ be countable

$\mathbb{R} = \underbrace{\mathbb{Q}}_{\text{countable}} \cup \underbrace{(\mathbb{R}\setminus\mathbb{Q})}_{\text{assumed to be countable}}$

$\Rightarrow$ $\mathbb{R}$ is countable $\Rightarrow\Leftarrow$

$\therefore$ Our assumption is wrong.

⊛ The set of all irrational numbers $(\mathbb{R}\setminus\mathbb{Q})$ is uncountable

★ $A_1, A_2, A_3, \ldots, A_m$: countable sets

m : finite $(m<\infty)$

$A_1 \cup A_2 \cup \ldots \cup A_m$: countable? $\underline{\underline{\text{Yes}}}$

A_1 $\quad A_2 \to A_2\setminus A_1$ $\quad A_3 \to A_3\setminus\{A_1, A_2\}$ $\quad \ldots \quad A_m \to A_m\setminus\{A_1, A_2, \ldots, A_{m-1}\}$

a_{11} — a_{21} — a_{13} — $\cdots$ — a_{1m}

a_{21} — a_{22} — a_{23} — $\cdots$ — a_{2m}

a_{31} — a_{32} — a_{33} — $\cdots$ — a_{3m}

$\vdots \qquad\qquad\qquad\qquad\qquad \vdots$

⊛ **Result**: If $A_1, A_2, \ldots, A_m$: countable sets, then their union $\underbrace{A_1 \cup A_2 \cup \ldots \cup A_m}_{\text{Finite union of countable sets}}$ is countable

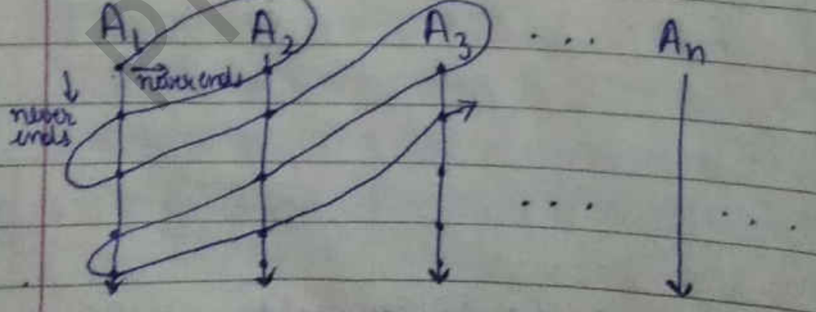
Finitely many countable sets

★ $\underbrace{A_1, A_2, A_3, \ldots, A_n, \ldots}$: countable sets

Countably many countable sets

$\bigcup_{n=1}^{\infty} A_n$: Is it countable? $\underline{\underline{\text{Yes}}}$

Countable union of countable sets



⊛ A finite union of countable sets is countable

⊛ A countable union of countable sets is countable.

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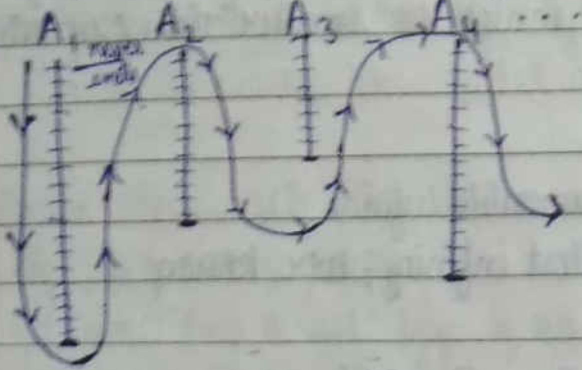
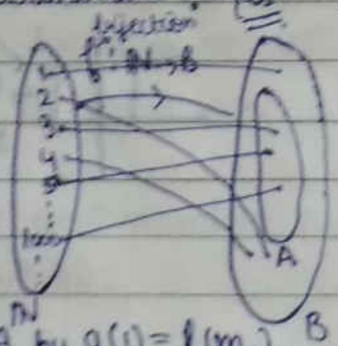
* countable union of finite sets? It is countable

$A_1, A_2, A_3, \ldots, A_n, \ldots$: finite sets



* suppose A(infinite set) $\subseteq B$, B : countable set

Is A countable? Yes



$m_1 = \min\{n \in \mathbb{N} : f(n) \in A\}$

$m_2 = \min\{n \in \mathbb{N} : f(n) \in A \setminus \{f(m_1)\}\}$

$m_3 = \min\{n \in \mathbb{N} : f(n) \in A \setminus \{f(m_1), f(m_2)\}\}$

$\vdots$

$m_k = \min\{n \in \mathbb{N} : f(n) \in A \setminus \{f(m_1), \ldots, f(m_{k-1})\}\}$

$\vdots$

$g : \mathbb{N} \to A$ by $g(1) = f(m_1)$

$g(2) = f(m_2)$

$\vdots$

$g(K) = f(m_K)$

* **Result:** The subsets of a countable set are countable or finite

DU 2016 Let X be a countable set. Suppose A is a subset of X, which is countable. Then $X \setminus A$

(a) is empty (b) is a finite set (c) can be uncountable

✓(d) can be countable infinite (e) is countable infinite

Sol^n (a) eg $X = \mathbb{N}$ $A = \{2, 3, 4, \ldots\}$

$X \setminus A = \{1\} \neq \phi$

(b) $X = \mathbb{Z}$ $A = \mathbb{N}$

$X \setminus A = \{0, -1, -2, \ldots\} \to$ not finite

(c) $X \setminus A \subseteq X$, so, can't be uncountable $\to$ either finite or countable

As X is countable & $X \setminus A$ is not finite, $\therefore X \setminus A$ is countable

(e) $X = \mathbb{N}$ $A = \{2, 3, 4, \ldots\}$, $X \setminus A = \{1\} \to$ not infinite.

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23/7/16

★ $\mathbb{R}$ $\mathbb{Q}$ larger ∞ & smaller ∞

uncountable Infinity countable infinity

Is there any "infinity" which is "smaller" than countable infinity? No

A: countable set

$B \subseteq A \to$ either finite or countable infinite

⊛ Countable infinity: smallest infinity; ever known

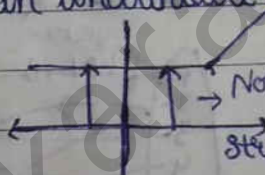
TIFR 2014

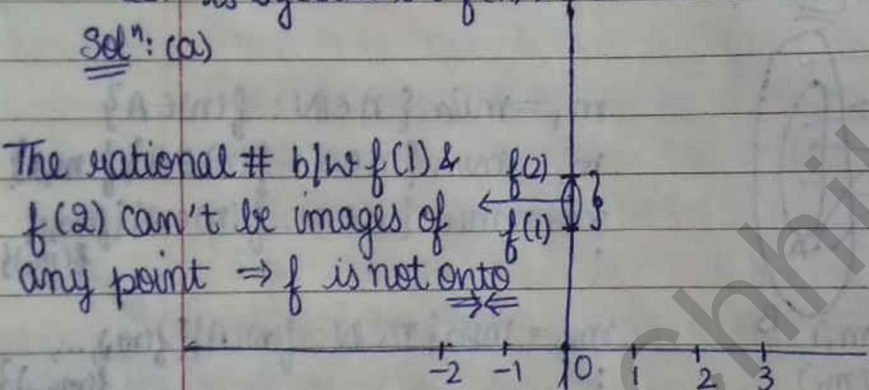
There exists a function $f: \mathbb{Z} \to \mathbb{Q}$ which

(a) is bijective & increasing (b) is onto & decreasing

✓(c) is bijective s.t. $f(n) \geq 0 \; \forall n \leq 0$ (d) has an uncountable image

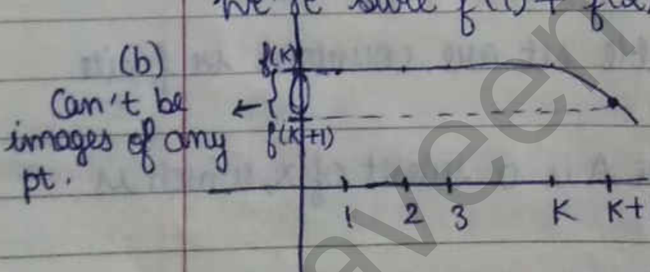
Sol^n: (a)





We're sure $f(1) \neq f(2)$, $f(1) < f(2)$ ($\because$ One-to-one $\Rightarrow$ St. inc.)

(b)



Let if possible, (b) be true

$\exists$ a $k \in \mathbb{Z}$ s.t. $f(k) \neq f(k+1)$

otherwise Range(f) would be singleton

(c) $\mathbb{Z}_0^- \subseteq \mathbb{Z} \to$ countable set

not finite $\Rightarrow$ countable set

$\mathbb{Z}^+ \to$ countable set

$\mathbb{Q} = \mathbb{Q}_0^+ \cup \mathbb{Q}^-$

countable set

$\mathbb{Z} = \mathbb{Z}^+ \cup \mathbb{Z}_0^-$ (C. set, C. set)

$\mathbb{Q} = \mathbb{Q}_0^+ \cup \mathbb{Q}^-$ (C. set, C. set)

bijection

$g: \mathbb{Z}_0^- \to \mathbb{Q}_0^+$

$h: \mathbb{Z}^+ \to \mathbb{Q}^-$

$f: \mathbb{Z} \to \mathbb{Q}$ by $f(n) = \begin{cases} g(n) & ; n \leq 0 \\ h(n) & ; n > 0 \end{cases}$

Bijection as f is one-to-one & onto

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Transcend → beyond human knowledge

$\mathbb{Z} = \mathbb{Z}^+ \cup \mathbb{Z}_0^-$
+ve   -ve including zero

$g(n) \neq h(n)\ \forall n$ as their co-domains are disjoint.

- **Algebraic Numbers**: Roots of Polynomials with integer coefficients, not all zeroes, are called algebraic numbers.

Q: Prove that each rational number is algebraic
Solⁿ: $p/q,\ p,q \in \mathbb{Z},\ q \neq 0$
$qx - p = 0$ has a solⁿ $p/q,\ q \neq 0$
$\because p/q$ is any rational number, $\therefore$ each rational number is algebraic

Q: Show $\sqrt{2}$ is an algebraic no.
Solⁿ: $x^2 - 2 = 0$ has a ~~root~~ solⁿ $\sqrt{2}$
$\therefore \sqrt{2}$ is an algebraic no.

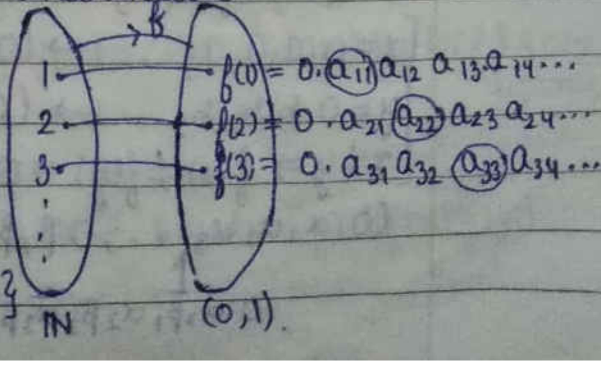
⊛ An irrational number can be algebraic but not all irrationals are algebraic.

⊛ π, e aren't algebraic
Transcendental nos: Real #s which are not algebraic

- **Liouville's Number**: $\sum_{k=1}^{\infty} 10^{-k!} \rightarrow$ Irrational #, Algebraic #

$= 10^{-1} + 10^{-2} + 10^{-6} + 10^{-24} + \ldots$
$0.1 + 0.01 + 0.000001 + \ldots = 0.110001000\ldots010\ldots$ (24th place)
No repitition & Non-terminating, $\therefore$ Irrational no.

⊛ **Result**: The open interval $(0,1)$ is uncountable.
Proof: Let if possible, $(0,1)$ be countable
Then $\exists$ a bijection $f: \mathbb{N} \rightarrow (0,1)$



$0 \leq a_i \leq 9$
or $a_i \in \{0, 1, \ldots, 9\}$

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Golden Ratio : $\frac{a+b}{a} = \frac{a}{b}$ → irrational & algebraic

$(0,1) = \{ f(1), f(2), f(3), \ldots \}$

$x = 0.b_1 b_2 b_3 \ldots \in (0,1)$

if $a_{11} = 1, b_1 = 2$; if $a_{11} \neq 1, b_1 = 1$ → $a_{11} \neq b_1$

if $a_{22} = 1, b_2 = 2$; if $a_{22} \neq 1, b_2 = 1$ $a_{22} \neq b_2$

$\vdots$ $a_{33} \neq b_3$

if $a_{nn} = 1, b_n = 2$; if $a_{nn} \neq 1, b_n = 1$

$$b_n = \begin{cases} 2 & ; \text{ if } a_{nn} = 1 \\ 1 & ; \text{ if } a_{nn} \neq 1 \end{cases}$$

$b_n \neq a_{nn} \ \forall \ n \in \mathbb{N}$

$x \neq f(1)$ ($\because$ 1st decimal pt. are distinct)

$x \neq f(2)$ ($\because$ 2nd decimal pt. are distinct)

$x \neq f(3)$ ($\because$ 3rd decimal pt. are distinct)

$\vdots$

$x \neq f(n) \ \forall \ n \in \mathbb{N} \Rightarrow\Leftarrow$

$\therefore$ Our assumption is wrong

$x \in (0,1)$ but $x \notin \{ f(1), f(2), \ldots \}$

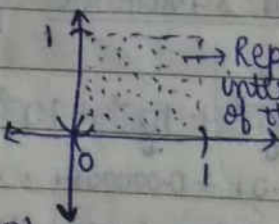
$\therefore f$ is not onto & hence not bijective

$\therefore (0,1)$ is uncountable.

$|(0,1)| = |\mathbb{R}|$

★ $(0,1)$ ← (0) 1 →

$(0,1) \times (0,1)$



<u>NEED</u>: $f : (0,1) \to (0,1) \times (0,1)$

one to one, onto

$g : (0,1) \to (0,1) \times (0,1)$ by $g(x) = (x, 0)$

One-to-One but not onto (Interior pts. has no image)

$0.a_1 a_2 a_3 a_4 a_5 \ldots \mapsto (0.a_1 a_3 a_5 \ldots, 0.a_2 a_4 a_6 \ldots) \leftarrow f(0.a_1 a_2 a_3 \ldots)$

$x = 0.a_1 a_2 a_3 \ldots \mapsto (0.a_1 a_3 a_5 \ldots, 0.a_2 a_4 a_6 \ldots)$

$y = 0.b_1 b_2 b_3 \ldots \mapsto (0.b_1 b_3 b_5 \ldots, 0.b_2 b_4 b_6 \ldots)$

$x \neq y \Rightarrow f(x) \neq f(y) \Rightarrow$ One-to-One

$(0.\alpha_1 \alpha_2 \alpha_3 \ldots, 0.\beta_1 \beta_2 \beta_3 \ldots)$ → Pick from the codomain i.e. $(0,1) \times (0,1)$

$\Rightarrow$ Onto

$0.\alpha_1 \beta_1 \alpha_2 \beta_2 \alpha_3 \beta_3 \ldots$

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$A=\{1,2,3\}$ $B=\{a,b\}$
$A\times B=\{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$

$\therefore |(0,1)| = |(0,1)\times(0,1)|$

TIFR 2012 True/False
There exists a bijection between $\mathbb{R}^2$ and the open interval $(0,1)$

Solⁿ: $(0,1)\sim\mathbb{R}$
$(0,1)\sim\underbrace{(0,1)\times(0,1)}_{\mathbb{R}\times\mathbb{R}}$? Yes $\Rightarrow (0,1)\sim\mathbb{R}^2$ $(\because (0,1)\sim\mathbb{R}$ & $(0,1)\sim\mathbb{R}$
$\Rightarrow (0,1)\times(0,1)\sim\mathbb{R}\times\mathbb{R})$

Equivalence relation

- **Transitive relation:** $A\sim B$ & $B\sim C$, then $A\sim C$
  $A\sim B \Rightarrow \exists$ a bijection $f: A\to B$
  $B\sim C \Rightarrow \exists$ a bijection $g: B\to C$
  $\Rightarrow g\circ f: A\to C$ (Bijection) $\Rightarrow A\sim C$

- **Symmetric relation:** $A\sim B \Rightarrow B\sim A$
  $A\sim B \Rightarrow \exists$ a bijection $f: A\to B$
  $\Rightarrow f^{-1}: B\to A$ is a bijection $\Rightarrow B\sim A$

- **Reflexive relation:** $A\sim A$
  $f: A\to A$ by $f(x)=x \to$ Bijection

★ $A\sim B$ & $C\sim D$, then $A\times C\sim B\times D$
$A\sim B\Rightarrow\exists$ a bijection $f:A\to B$
$C\sim D\Rightarrow\exists$ a bijection $g:C\to D$
NEED: $h: A\times C\to B\times D$
$h(x,y)=(f(x),g(y))$

h is one-to-one
$h(x_1,y_1)=h(x_2,y_2)\Rightarrow (f(x_1),g(y_1))=(f(x_2),g(y_2))$
$\Rightarrow f(x_1)=f(x_2)$ & $g(y_1)=g(y_2)\Rightarrow x_1=x_2$ & $y_1=y_2$ ($\because f$ & g are bijective)
$\Rightarrow (x_1,y_1)=(x_2,y_2)$

h is onto
Let $(f(x),g(y))\in B\times D\Rightarrow\exists\,(x,y)\in A\times C$ s.t. $h(x,y)=(f(x),g(y))$
$\therefore h$ is a bijection

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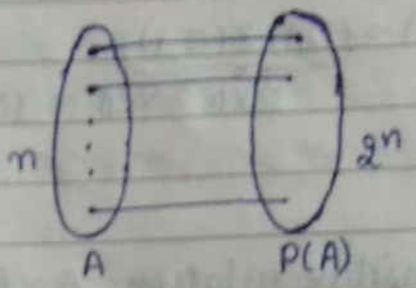
* $|\mathbb{R}| = |\mathbb{R}^2|$ (as $(0,1) \sim \mathbb{R}$ & $(0,1) \sim \mathbb{R}^2$)
$= |\mathbb{R}^3| = |\mathbb{R}^4| = \dots = |\mathbb{R}^m|$, where $m \in \mathbb{N}$

Bijection $\leftarrow f : (0,1) \to \underbrace{(0,1) \times (0,1) \times (0,1)}_{\sim \mathbb{R}^3}$

30/7/16

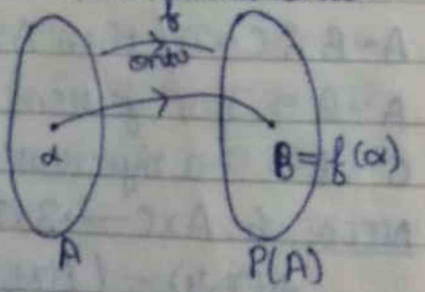
★ A: finite set
$|A| = n < \infty$
$|P(A)| = 2^n$
$f : A \to P(A)$



Is f a onto function? ~~Yes~~ No
Whatever be $|\text{Range}(f)| \le n$
$2^n > n$
$\text{Range}(f) \subsetneq P(A) \ \forall f$
If A is a finite set, then there exists no onto function
$f : A \to P(A)$
What about infinite sets? Yes

- Cantor's Theorem: If A is a set, then there exists no function $f : A \to P(A)$, that is onto.

Proof: Let if possible, there exist $f : A \to P(A)$ which is onto.
$\underbrace{B = \{x \in A : x \notin f(x)\}}_{\text{subset of } A}$



$B \subseteq A \Rightarrow B \in P(A)$
Now, f is onto, $\therefore \exists$ some $\alpha \in A$ such that $f(\alpha) = B$
$f(\alpha) = \{x \in A : x \notin f(x)\}$
Is $\alpha \in B$?
Let $\alpha \in B$, then $\alpha \notin f(\alpha)$ i.e. $\alpha \notin B \Rightarrow\Leftarrow$
Let $\alpha \notin B$, then $\alpha \in f(\alpha)$ i.e. $\alpha \in B \Rightarrow\Leftarrow$
$\to$ So, $\exists$ no func^n $f : A \to P(A)$ which is onto

What about one-to-one functions?
$\left\{ \begin{array}{l} f : A \to P(A) \ \text{one-to-one} \\ x \longmapsto \{x\} \end{array} \right.$
Impossible $\leftarrow f : A \to P(A)$ onto

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CONTINUUM HYPOTHESIS - There is no infinity b/w them

⊛ set containing all the sets in the universe?
NO, such set exists, eg: $|\mathbb{R}| \leq |P(\mathbb{R})| < |P(P(\mathbb{R}))| < \dots$

⊛ $|\mathbb{R}| < |P(\mathbb{R})| < |P(P(\mathbb{R}))| < \dots$

⊛ There doesnot exist any "biggest infinity."

⊛ $P(\overset{\text{countable infinity}}{\mathbb{N}})$ is an uncountable set $(\because |\mathbb{N}| < |P(\mathbb{N})|)$

⊛ $|\text{Finite sets}| < |\mathbb{N}| < \underbrace{|\mathbb{R}| < |P(\mathbb{R})| < |P(P(\mathbb{R}))| < \dots}_{\text{uncountable infinities}}$
$\aleph_0 < \aleph_1 < \aleph_2$

⊛ $\underset{|\mathbb{N}| = \aleph_0}{|P(\mathbb{N})|} = |\mathbb{R}| = C \to$ Continuum Hypothesis $\mathbb{R} \to$ smallest uncountable infinity

CSIR Which of the following set is (are) uncountable
✓(a) $\{f \mid f : \mathbb{N} \to \{1,2\}\}$ (b) $\{f \mid f : \{1,2\} \to \mathbb{N}\}$
(c) $\{f \mid f : \{1,2\} \to \mathbb{N}, f(1) \leq f(2)\}$
✓(d) $\{f \mid f : \mathbb{N} \to \{1,2\}, f(1) \leq f(2)\}$

Sol^n: (a) Claim: $|A| = |P(\mathbb{N})| \to$ uncountable

$\phi : P(\mathbb{N}) \to A$ by $\phi(X) = g$

$X \subseteq A$, $g : \mathbb{N} \to \{1,2\}$
$X \in P(A)$ $g(\alpha) = \begin{cases} 1 & \text{if } \alpha \in X \\ 2 & \text{if } \alpha \notin X \end{cases}$

To show ϕ : one-to-one
To show: $X \neq Y \Rightarrow \overset{g}{\phi(X)} \neq \overset{h}{\phi(Y)}$
Without loss of generality, $X \not\subseteq Y$, ~~$Y \not\subseteq X$~~ $X \subseteq Y$
There exists $\alpha \in X$ but $\alpha \notin Y$
$g(\alpha) = 1$, $h(\alpha) = 2 \Rightarrow \phi(X) \neq \phi(Y)$

To show: ϕ : onto
Pick an $g \in A$
$g : \mathbb{N} \to \{1,2\}$
$\phi(\{n \in \mathbb{N} : g(n) = 1\}) = g$

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$K\aleph_0 = \aleph_0$, K: finite

$\aleph_0^K = \dots = \aleph_0^4 = \aleph_0^3 = \aleph_0^2 = \aleph_0 \quad K < \infty$

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Date / /

If $X = \mathbb{N}$, then $\phi(X) = g$, where $g \equiv 1$

If $\cancel{X \neq \mathbb{N}}$ $X = \phi$, then $\phi(X) = g$, where $g \equiv 2$ or $2^{\aleph_0} = c$ is cardinality of $\mathbb{R}$ which uncountable

Claim: If A and B: countable sets, then $A \times B$ is countable

$A = \{a_1, a_2, a_3, \dots\}$, $B = \{b_1, b_2, b_3, \dots\}$

$$\begin{array}{cccc}
(a_1,b_1) & (a_1,b_2) & (a_1,b_3) & \dots \\
(a_2,b_1) & (a_2,b_2) & (a_2,b_3) & \dots \\
(a_3,b_1) & (a_3,b_2) & (a_3,b_3) & \dots
\end{array}$$

If A, B and C: countable

$A \times B \times C = (A \times B) \times C$

✫ $A_1, A_2, A_3, \dots, A_n$: countable sets

$A_1 \times A_2 \times A_3 \times \dots \times A_n$: countable sets, $n < \infty$

(Using Mathematical Induction)

⊛ A finite cartesian product of countable sets is countable.

Proof: Using induction,

– $A_1, A_2, A_3, \dots, A_n \dots$ Infinite many countable sets

$A_1 \times A_2 \times A_3 \times \dots \times A_n \times \dots$ may not be countable

(b) $B = \{f \mid f: \{1,2\} \to \mathbb{N}\}$

Claim: $|B| = |\mathbb{N} \times \mathbb{N}| = \aleph_0 \times \aleph_0 = \aleph_0^2 = \aleph_0 \to$ countable

~~$g \mapsto (4,5)$~~

~~$g(1) = 4 \quad g(2) = 5$~~

~~$\phi(f) = (f(1), f(2))$~~

~~$\{f \mid f: \mathbb{N} \to \{1,2\}\}$~~

~~$2^{\aleph_0} = c$~~

~~So, it is uncountable.~~

To show: ϕ is 1:1, $\phi: B \to \mathbb{N} \times \mathbb{N}$

$\phi(f) = (f(1), f(2))$

To show: ϕ is onto

$g \mapsto (4,5)$

$g(1) = 4 \quad g(2) = 5$

Since it is one-to-one correspondence

$\Rightarrow |B| = |\mathbb{N} \times \mathbb{N}|$

($\mathbb{N}$ is infinitely countable $\Rightarrow$ it is countable)

(c) $C \subseteq B$

~~B is countable~~

$C = \{f \mid f: \{1,2\} \to \mathbb{N}, f(1) \le f(2)\}$

$\aleph_0 \quad \aleph_0$

1 2 3 $f(1)$ choices for $f(2)$

Or $\left.\begin{array}{l}C \text{ is a subset of countable set} \\ \& \ C \text{ is infinite}\end{array}\right\} \Rightarrow C$ is countable

(d) $D \subseteq A$, $D = \{f \mid f: \mathbb{N} \to \{1,2\}, f(1) \le f(2)\}$

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$2^{\mathbb{N}}$: set of all sequences with terms 0 or 1, 1 or 2, ...

A is uncountable

$f(1) = 1$, $f(2) = 1, 2$

$f(1) = 2$, $f(2) = 2$

$\{1, 2\} \to 3$, $3 \to 2$, $4 \to 2$ ...

$3 \times 2^{\aleph_0} = 3 \times \mathfrak{c} = \mathfrak{c}$ $(\because k\,\mathfrak{c} = \mathfrak{c})$

So, it is uncountable

TIFR 2013 True / ~~False~~

Let S be the set of all sequences $\{a_1, a_2, \ldots, a_n, \}$, where each entry a_i is either 0 or 1. Then S is countable.

Sol^n: ϕ : one-to-one

There exist $\phi : S \to P(\mathbb{N})$

$\phi(\{a_1, a_2, \ldots, a_n, \ldots\}) = \{m \in \mathbb{N} : a_m = 1\}$

So, it is uncountable.

DU 2015 Which of the following sets are not countable

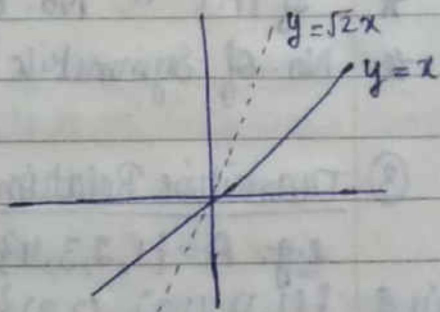
(a) $\mathbb{Z}$ (b) $\{1,2\}^{\mathbb{N}}$, the set of all sequences with terms 1 or 2.

(c) $\mathbb{Q}$ (d) $\sqrt{2}\,\mathbb{Q}$

Sol^n: (d) To show: one-to-one correspondence to $\mathbb{Q}$

$\phi : \mathbb{Q} \to \sqrt{2}\mathbb{Q}$

$\phi(x) = \sqrt{2}\, x$



one-one

$x_1 \neq x_2 \Rightarrow \phi(x_1) \neq \phi(x_2)$

Onto Pick $z \in \sqrt{2}\mathbb{Q}$

$\frac{z}{\sqrt{2}} \longmapsto z$

Q: A: Subset of $A \times B$: relation from A to B

$A = \{1, 2, 3\}$ $B = \{a, b\}$

(a) $\{(1, b), (3, a)\} \subseteq A \times B$

(b) $\{(4, a), (2, b)\} \nsubseteq A \times B$

(c) $\{(1, c), (2, a)\} \nsubseteq A \times B$

(d) $\{(b, 1), (a, 3)\} \subseteq B \times A$ (relation from B to A)

$A = B$

relation from A to A

Or relation on A

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$2^{\mathbb{N}}$: set of all sequences with terms 0 or 1, 1 or 2, ...

A is uncountable

$f(1) = 1$, $f(2) = 1, 2$

$f(2) = 2$, $f(2) = 2$

$\{1, 2\}$ $\to$ 3, 3 $\to$ 2, 4 $\to$ 2 ...

$3 \times 2^{\aleph_0} = \cancel{3 \times} \mathfrak{c} = \mathfrak{c}$ $(\because k\,\mathfrak{c} = \mathfrak{c})$

So, it is uncountable

TIFR 2013 True / False ✓

Let S be the set of all sequences $\{a_1, a_2, \ldots, a_n, \ldots\}$, where each entry a_i is either 0 or 1. Then S is countable.

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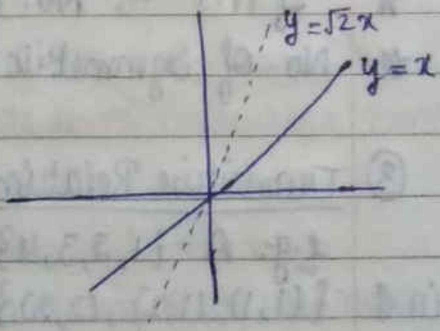

one-one

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A = B

relation from A to A

or relation on A

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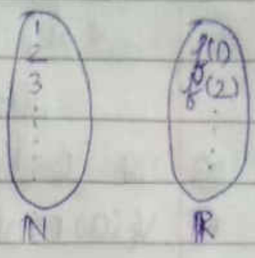
13/3/16

Sequences

- $f: \mathbb{N} \to \mathbb{R}$
  A sequence is a function whose domain is $\mathbb{N}$.
  $\langle \underset{\text{terms}}{f(1)}, f(2), f(3), \ldots \rangle$



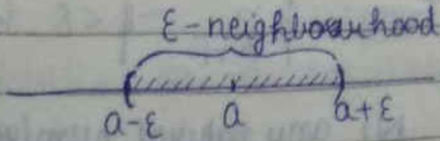
Sequences		Sets	
1.	Terms can be repeated	1.	Elements can't be repeated.
2.	Order matters	2.	Order doesn't matter

* $\langle f(1), f(2), f(3), \ldots \rangle$ or
$\langle f_1, f_2, f_3, \ldots \rangle$
f_n: image of n
$\langle a_1, a_2, \ldots \rangle$
$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right), \left(\frac{n+1}{n}\right)_{n=1}^{\infty}, (a_n)$, where $a_n = 2^n$

- <u>Convergence of sequence:</u>
  (a_n) approaches 'a' $\to$ Meaning?
  n is just act as a time, as soon as n increases time increases and we get nearer to a.
  <u>Given</u>: $\varepsilon > 0$ Seek: $N \in \mathbb{N}$
  $|a_n - a| < \underset{\text{how much get nearer}}{\varepsilon}$ $\forall\ n \geq N \to$ time when terms of seq. get more nearer to a

$\to$ <u>Definition</u>: A sequence (a_n) is said to converge to 'a' if for a given $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that
$$|a_n - a| < \varepsilon \ \forall\ n \geq N$$

- $a \in \mathbb{R}$, $\varepsilon > 0$
  $V_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\}$



We make ε i.e. radius smaller & smaller such that all terms get inside in its neighbourhood.

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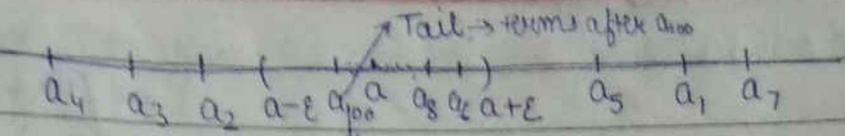
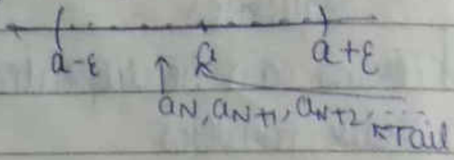


After a_{100} all terms are in ε - neighborhood of 'a'

- N: instant after which sequence enters $V_\varepsilon(a)$, never to leave
- $V_\varepsilon(a)$ contains all but finitely many terms of (a_n)
- If ε is made smaller, then N may be higher.
- N depends on the choice of ε
- At most N-1 terms aren't in the ε - neighborhood of 'a'.



Q- Show (a_n), where $a_n = \frac{1}{\sqrt{n}}$ converges to zero.

Sol^n: $\varepsilon = \frac{1}{10}$

GOAL: $|a_n - 0| < \frac{1}{10}$

i.e. $\frac{1}{\sqrt{n}} < \frac{1}{10}$

i.e. $n > 100$ → Set $N = 101$

If $\varepsilon = \frac{1}{100}$, then $n > 10000$ → $N = 10001$

Duel of Challenge (ε) & Response (N)

GOAL: $\left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon$ i.e. $\frac{1}{\sqrt{n}} < \varepsilon$ i.e. $n > \frac{1}{\varepsilon^2}$

Take $N = \left[\frac{1}{\varepsilon^2}\right] + 1$

$\frac{1}{\varepsilon^2}$ need not be natural no.

$\therefore (a_n) \longrightarrow 0$ i.e. $\lim_{n\to\infty} a_n = 0$

Q- Show $\left(\frac{n}{n+1}\right) \longrightarrow 1$

Sol^n: Given: $\varepsilon > 0$

GOAL: $\left|\frac{n}{n+1} - 1\right| < \varepsilon$ i.e. $\frac{1}{n+1} < \varepsilon$ i.e. $n+1 > \frac{1}{\varepsilon}$ i.e. $n > \frac{1}{\varepsilon} - 1$

N: any natural number greater than $\frac{1}{\varepsilon} - 1$

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Let the no. be $a \to x$
$\downarrow$
a ✓ symbol.

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Date / /

V. Imp.

Q- Describe how would we demonstrate the following statements invalid.
① At each college of the United States, there is a student who is at least seven feet tall.
② For each college of the United States, there is a professor who give every students grades either A or B.
③ There exists a college in the United States, where each student is at least six feet tall

Sol^n ① There ~~exists~~ is a college of the U.S., where each student ~~is who~~ is less than (at most) seven feet tall.
② There ~~exist~~ is a college of the U.S, where each professor gives at least one student grades neither A nor B.
③ At each college in the US, there is a student who is less than (at most) six feet tall.
④ ~[For all ...] = there exists a ...

TIFR 2014 Let A, B, C be subsets of $\mathbb{R}$. What is the negation of the following statement?
For each $\varepsilon > 1$, there exists $a \in A$, $b \in B$ such that for all $c \in C$, we have $|a-c| < \varepsilon$ & $|b-c| > \varepsilon$
(a) There exists an $\varepsilon \le 1$, such that for all $a \in A$, $b \in B$, there exists a $c \in C$, such that $|a-c| \geq \varepsilon$ & $|b-c| \le \varepsilon$
(b) There exists an $\varepsilon \le 1$, such that for all $a \in A$, $b \in B$, there exists a $c \in C$, such that $|a-c| \geq \varepsilon$ or $|b-c| \le \varepsilon$
(c) There exists an $\varepsilon > 1$, such that for all $a \in A$, $b \in B$, there exists a $c \in C$, such that $|a-c| \geq \varepsilon$ & $|b-c| \le \varepsilon$
✓(d) There exists an $\varepsilon > 1$, such that for all $a \in A$, $b \in B$ there exists a $c \in C$ such that $|a-c| \geq \varepsilon$ or $|b-c| \le \varepsilon$

- $(a_n) \to a$
For each $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ s.t. $|a_n - a| < \varepsilon \ \forall\ n \geq N$
$(a_n) \not\to a \longrightarrow$ Negation
What does it mean when if we say that (a_n) doesnot converge to a?

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$1 - 2$ — minus operation; -2 — negative 2, not operation

Definition of non-convergence of sequence: "There exists an $\varepsilon > 0$, such that for all $N \in \mathbb{N}$, there exists an $M \geq N$ s.t. $|a_M - a| \geq \varepsilon$."

Q- Show that $\left(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{-1}{8}, \ldots\right)$ doesnot converge to zero.

Sol^n: If we take $\varepsilon = \frac{1}{16}$, no N works!

Q- Show that $\left(\frac{1}{8}, 0, \frac{-1}{8}, 0, \frac{1}{8}, 0 \, \frac{-1}{8}, \ldots\right)$ doesnot converge to zero

Sol^n: If we take $\varepsilon = \frac{1}{16}$, find some suitable N.
there is no such N

★ To show: $\left(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{-1}{8}, \ldots\right) \not\to l$, for any $l \in \mathbb{R}$

If $l \neq \pm\frac{1}{8}$, then
$\varepsilon := \min\left\{ \left|l - \left(-\frac{1}{8}\right)\right|, \left|\frac{1}{8} - l\right| \right\}$

If $l = \frac{1}{8}, \frac{-1}{8}$, then $\varepsilon = \dfrac{\left|\frac{1}{8} - \left(-\frac{1}{8}\right)\right|}{2} = \frac{1}{8}$

Q- Argue that the sequence $(1, 0, 1, 0, 0, 1, 0, 0, 0, 1, \ldots)$ doesn't converge to zero (a) For what $\varepsilon > 0$, we get a response N?
(b) For what $\varepsilon > 0$, we don't get any response N?

Sol^n: (a) $(0-2, 0+2) \rightarrow \varepsilon = 2, N = 1$
1 go outside

(b) If $\varepsilon = 1$, then we don't get any N as 1 go outside from it

- For $\varepsilon > 0$, we get suitable $N = N_0$; any $N' > N_0$ also a suitable N
- $\varepsilon_0 > 0$, suitable N. Suppose $\varepsilon' > \varepsilon_0$: $(a-\varepsilon', a-\varepsilon_0, a, a+\varepsilon_0, a+\varepsilon')$. Same N would work

★ A: bounded set $\Rightarrow \exists$ some $M > 0$ s.t. $|x| < M \;\forall x \in A$

⊛ To check boundedness, we measure range (vertically not horizontally)

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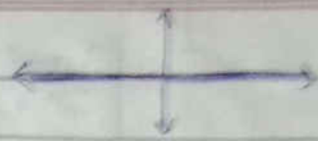
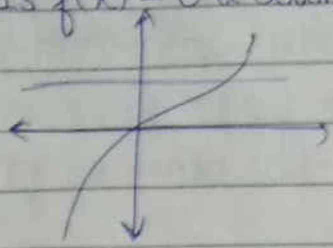
Function is bounded, if range is bounded.

$|x| = \max.\{x, -x\}$

$||a|-|b|| \le |a-b| \to \max.\{|a|-|b|, |b|-|a|\}$ $\to |a|-|b| \le |a-b|$, $|b|-|a| \le |a-b|$

★ Is $f(x) = 0$ a bounded function? Yes





No, range is not bounded in this case so, function is not bounded.

★ $\langle 1, 0, -1, 0, 1, \ldots \rangle \to$ Range $= \{1, 0, -1\}$

• Definition: A sequence (a_n) is said to be bounded if there exists an $M>0$ such that $|a_n| < M \ \forall n \in \mathbb{N}$

$-M < a_n < M \ \forall n \in \mathbb{N}$

⊛ Result: A convergent sequence is bounded.

Proof: Let $(a_n) \to a$, $\varepsilon = 1$, we get an $N \in \mathbb{N}$ s.t. $|a_n - a| < 1 \ \forall n \in \mathbb{N}$ — ①

$|a_n| - |a| \le |a_n - a|$ — ②

① & ② $\Rightarrow |a_n| - |a| < 1 \ \forall n \ge N$

$\Rightarrow |a_n| < |a| + 1 \ \forall n \ge N$

Let $M = \max.\{|a_1|, |a_2|, |a_3|, \ldots, |a_{N-1}|, |a|+1\}$

$\Rightarrow |a_n| \le M$

It contains all the terms after N

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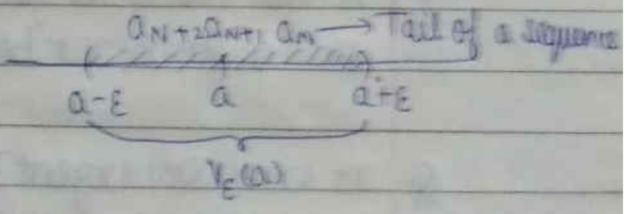
• $(a_n) \to a$

given: $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$|a_n - a| < \varepsilon \ \forall n \ge N$



Q- Show $\lim_{n\to\infty} \dfrac{3n+1}{2n+2} = \dfrac{3}{2}$

Sol^n:

$a_n = \dfrac{3n+1}{2n+2}$

given $\varepsilon > 0$

GOAL: $\left|a_n - \dfrac{3}{2}\right| < \varepsilon$

Seek some suitable N

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$$\left|\frac{3n+1}{2n+2} - \frac{3}{2}\right| < \varepsilon$$

i.e. $\left|\frac{-1}{n+1}\right| < \varepsilon$

i.e. $n+1 > \frac{1}{\varepsilon}$ i.e. $n > \frac{1}{\varepsilon} - 1$

Choose any natural number greater than $\frac{1}{\varepsilon} - 1$.
$\downarrow$ N

Q- $\langle c, c, c, c, \ldots \rangle \longrightarrow c$. Show it.

Soln: Given: $\varepsilon > 0$ $\quad a_n = c \ \forall n \in \mathbb{N}$

GOAL: $|a_n - c| < \varepsilon$

i.e. $|c - c| < \varepsilon$

i.e. $0 < \varepsilon$

Choose $N = 1$.

★ Eventually constant sequences: $\langle \underbrace{c \ldots\ldots}_{\text{can be anything}}, \underbrace{c, c, c \ldots}_{\text{const.}} \rangle$

⊛ We can ignore ~~beginning~~ first finitely many terms of a sequence in regard to its convergence.

⊛ "Finite" word is never used, we can use "finitely many" or "a finite number of".

Q- Is $\langle n \rangle$ convergent? No

Soln: $\langle n \rangle$ is not bounded, and consequently it is not convergent.

Result:

⊛ $\langle a_n \rangle, \langle b_n \rangle$: sequences

$\lim a_n = a$, $\lim b_n = b$, $c \in \mathbb{R}$

Then

① $\lim ca_n = ca$ ② $\lim (a_n + b_n) = a + b$

③ $\lim (a_n b_n) = ab$ ④ $\lim \frac{a_n}{b_n} = \frac{a}{b}$, if $b \neq 0$

This is known as Algebraic limit ~~of~~ Theorem.

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$|x+y| \le |x|+|y|$

Proof ① Given: $\varepsilon>0$
Case I: $c \neq 0$
GOAL: $|ca_n - ca| < \varepsilon$
i.e. $|c||a_n - a| < \varepsilon$

If we make $|a_n - a| < \frac{\varepsilon}{|c|}$, our work is done.

Since $(a_n) \to a$, there exists an $N \in \mathbb{N}$ s.t.
$$|a_n - a| < \frac{\varepsilon}{|c|} \quad \forall n \ge N.$$
$\Rightarrow |c|\,|a_n - a| < \varepsilon \;\forall n \ge N$
So, our goal is accomplished.
Case II: $c = 0$
It is obvious

② Given: $\varepsilon > 0$
GOAL: $|(a_n+b_n) - (a+b)| < \varepsilon$
i.e. $|(a_n-a)+(b_n-b)| \le |a_n-a|+|b_n-b| < \varepsilon$ —— Ⓐ
$(a_n) \to a \Rightarrow \exists$ some $N_1 \in \mathbb{N}$ s.t. $|a_n - a| < \frac{\varepsilon}{2} \;\forall n \ge N_1$
$(b_n) \to b \Rightarrow \exists$ some $N_2 \in \mathbb{N}$ s.t. $|b_n - b| < \frac{\varepsilon}{2} \;\forall n \ge N_2$
Set $N = \max.\{N_1, N_2\}$
$$\left.\begin{array}{l} |a_n - a| < \frac{\varepsilon}{2} \\ |b_n - b| < \frac{\varepsilon}{2} \end{array}\right\} \forall n \ge N$$
$\therefore |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \;\forall n \ge N$
i.e. $|a_n - a| + |b_n - b| < \varepsilon \;\forall n \ge N$
From Ⓐ, $|a_n + b_n - (a+b)| < \varepsilon \;\forall n \ge N$.

③ Given: $\varepsilon > 0$
GOAL: $|a_n b_n - ab| < \varepsilon$
$$|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab|$$
$$\le |a_n b_n - ab_n| + |ab_n - ab|$$
$$\le |b_n||a_n - a| + |a||b_n - b| \quad \text{—— Ⓑ}$$
Now, (b_n) is cgt. $\Rightarrow (b_n)$ is bdd.
$\Rightarrow$ There exists an $M > 0$ s.t. $|b_n| < M \;\forall n \in \mathbb{N}$
Ⓑ $\Rightarrow |a_n b_n - ab| < M|a_n - a| + |a||b_n - b|$ —— Ⓒ

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$\frac{|a|}{|a|+1} < 1$

If $a=0$, then $0<1$

$||x|-|y|| \le |x-y|$

$\Rightarrow |x|-|y| \le |x-y|$

$|y|-|x| \le |x-y|$

$[\because |x| = \max\{x,-x\}]$

$(b_n) \to b \Rightarrow$ There exists an $N_1 \in \mathbb{N}$ s.t. $|b_n - b| < \frac{\varepsilon}{2(|a|+1)}$ $\forall n \ge N_1$

$(a_n) \to a \Rightarrow$ There exists an $N_2 \in \mathbb{N}$ s.t. $|a_n - a| < \frac{\varepsilon}{2M}$ $\forall n \ge N_2$

$N := \max\{N_1, N_2\}$

$$\left.\begin{array}{l} |b_n - b| < \frac{\varepsilon}{2(|a|+1)} \\ \\ |a_n - a| < \frac{\varepsilon}{2M} \end{array}\right\} \forall n \ge N$$

(C) $\Rightarrow$ $M|a_n - a| + |a||b_n - b| < \cancel{M}\cdot\frac{\varepsilon}{2\cancel{M}} + |a|\cdot\frac{\varepsilon}{2(|a|+1)}$ $\forall n \ge N$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ if $n \ge N$

i.e. $|a_nb_n - ab| < \varepsilon$ $\forall n \ge N$

(4) Given: $\varepsilon > 0$

It is sufficient to show that $\lim \frac{1}{b_n} = \frac{1}{b}$

GOAL: $\left|\frac{1}{b_n} - \frac{1}{b}\right| < \varepsilon$

$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b_n - b|}{|b_n||b|}$ — (1)

If we use $|b_n| < M$
then $\frac{1}{|b_n|} > \frac{1}{M}$
which is not appropriate
So, we can't do this

$(b_n) \to b$, There exists an $N_1 \in \mathbb{N}$ s.t. $|b_n - b| < \underset{\frac{|b|}{2}}{\varepsilon}$ $\forall n \ge N_1$

$||b_n| - |b|| \le |b_n - b|$

$\Rightarrow$ $||b_n| - |b|| < \varepsilon$ $\forall n \ge N_1$

$\Rightarrow$ $|b_n| - |b| < \varepsilon \Rightarrow |b_n| < \varepsilon + |b| \Rightarrow$ Not required (Take $M = \varepsilon + |b|$ in (3))

$|b| - |b_n| < \underset{\frac{|b|}{2}}{\varepsilon} \Rightarrow |b_n| > |b| - \underset{\frac{|b|}{2}}{\varepsilon} \Rightarrow |b_n| > \frac{|b|}{2}$ $\forall n \ge N_1$

(1) $\Rightarrow$ $\left|\frac{1}{b_n} - \frac{1}{b}\right| < \frac{|b_n - b|}{|b_n||b|} < \frac{|b_n - b|}{\frac{|b|}{2}\,|b|}$ if $n \ge N_1$

$(b_n) \to b \Rightarrow \exists$ some $N_2 \in \mathbb{N}$ s.t. $|b_n - b| < \frac{\varepsilon}{2}|b|^2$ $\forall n \ge N_2$

Take $N = \max\{N_1, N_2\}$

$\frac{|b_n - b|}{\frac{|b|\,|b|}{2}} < \varepsilon$ $\forall n \ge N$.

- $(a_n), (b_n)$: Sequences in $\mathbb{R}$

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$(a_n) \to a$, $(b_n) \to b$, $c \in \mathbb{R}$

Then. ① if $a_n \geq 0 \ \forall n \in \mathbb{N}$, then $a \geq 0$

② if $a_n \geq b_n \ \forall n \in \mathbb{N}$, then $a \geq b$

③ if $a_n \geq c \ \forall n \in \mathbb{N}$, then $a \geq c$.

Proof: ① Let, if possible, $a < 0$

a_n 0

Take $\varepsilon = -\frac{a}{2}$ or $\frac{|a|}{2}$ $\left(-\frac{a}{2} > 0 \text{ as } a < 0, \text{ so } \varepsilon > 0\right)$

$(a_n) \to a \Rightarrow$ There exists an $N \in \mathbb{N}$ s.t. $a_n \in V_\varepsilon(a) \ \forall n \geq N$

So, we get a contradiction — It contains -ve #s.

Hence, $a \geq 0$

② $a_n \geq b_n \Rightarrow a_n - b_n \geq 0$

Let $c_n = a_n - b_n$.

$\lim c_n = \lim (a_n - b_n) = \lim (a_n + (-1)b_n) = \lim a_n + \lim (-1) b_n$

$= a - b$.

$c_n \geq 0 \Rightarrow \lim c_n \geq 0 \Rightarrow a - b \geq 0 \Rightarrow a \geq b$

20/8/16

Exercise - 2.2

2.2.1)(a) $\lim \frac{1}{(6n^2+1)} = 0$

Given: $\varepsilon > 0$. It is sufficient to assume $0 < \varepsilon < 1$

$a_n = \frac{1}{6n^2+1}$

GOAL: $|a_n - 0| < \varepsilon$

i.e. $\frac{1}{6n^2+1} < \varepsilon$

i.e. $6n^2 + 1 > \frac{1}{\varepsilon}$

i.e. $n > \sqrt{\frac{1/\varepsilon - 1}{6}}$ (For $\varepsilon > 1$, root becomes non-real, so we can find $N \in \mathbb{N}$ for $0 < \varepsilon < 1$ and by ⊛ same N works for $\varepsilon > 1$)

⊛ Demand: $|a_n - l| < \varepsilon_2$

$|a_n - l| < \varepsilon_1$

For $\varepsilon = \varepsilon_1$, we get desired N

For $\varepsilon = \varepsilon_2$, $\varepsilon_2 > \varepsilon_1$, same N works.

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$\lfloor \; \rfloor \to |a_n - l| < \varepsilon \; \forall n \geq 1$

So, we let $0 < \varepsilon < 1$

2.2.6) (a) larger (b) larger

2.2.2) Quantifiers ← For all $\varepsilon > 0$ & there exists an $N \in \mathbb{N}$ s.t. $|a_n - l| < \varepsilon \; \forall n \geq N$], converg seq

There exists an $\varepsilon > 0$ such that for all $N \in \mathbb{N}$, we have $|a_n - l| < \varepsilon \; \forall \; n \geq N_1$] → vercongent sequences.

(Not always) ← Are convergent sequences vercongent? e.g. Const. seq.

⊛ Vercongent sequences must be bounded ($\because |a_n - l| < \varepsilon \; \forall n \geq 1$ i.e. $\to l$ $l - \varepsilon < a_n < l + \varepsilon$) [Prove it!]

Vercongent but divergent sequence → $\langle 1, -1, 1, -1, \ldots \rangle$

$\varepsilon = 3$, $l = 1$, $|a_n - 1| < \varepsilon$

2.2.5) (a) $a_n = \left[\left[\frac{1}{n}\right]\right] \to \langle 1, 0, 0, 0, \ldots \rangle$ (n=1, n=2, n=3, n=4) → Eventually constant sequence → Always convergent

$\therefore a_n = \left[\left[\frac{1}{n}\right]\right]$ converges to zero.

So, $N = 2$ works for all $\varepsilon > 0$

GOAL: $|a_n - 0| < \varepsilon$ i.e. $0 < \varepsilon$

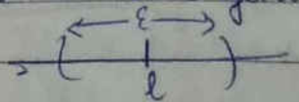
This goal is trivially true.

★ $a_n = \left[\left[\frac{1}{(2n)^3}\right]\right] \to \langle 0, 0, 0, \ldots \rangle$ $N = 1$ works for all $\varepsilon > 0$

(b) $a_n = \left[\left[\frac{10+n}{2n}\right]\right] = \langle 5, 3, 2, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, \ldots \rangle$

$N = 11$ works for all $\varepsilon > 0$.

★ Bounded but not vercongent? Not possible as we get ε



★ Vercongent & Convergent sequence → $\langle c, c, c, \ldots \rangle$

2.2.7) (a) $\lim a_n = \infty$

How to define it?

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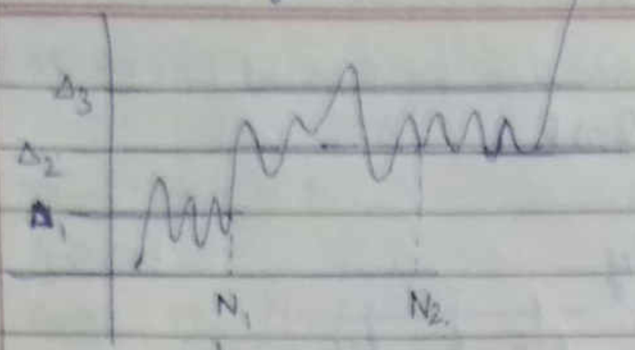
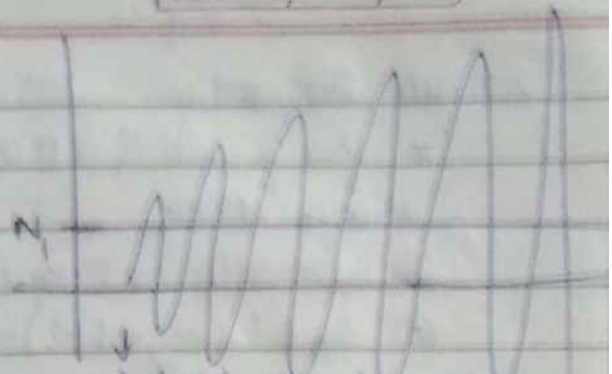
Oscillatory sequence $\leftarrow \lim \langle (-1)^n n \rangle \neq \infty$
(goes above but also comes below so, direction is not fixed).



can be categorize in $\lim a_n = \infty$



can't decide the direction. for any N_1, the graph doesn't go beyond N_1 always, so can't be categorize in $\lim a_n = \infty$

- Definition: We write $\lim a_n = \infty$, if for any $\Delta \in \mathbb{R}$ $\exists$ an $N \in \mathbb{N}$ s.t. $a_n > \Delta \ \forall n \geq N$
  (a_n) diverges to ∞ or converges? to ∞ (jo likha h vo nhi hota)
  e.g: $\langle 1, 2, 3, \ldots \rangle$, $\langle 2^n \rangle$
- We write $\lim a_n = -\infty$, if for any $\Delta \in \mathbb{R}$ $\exists$ an $N \in \mathbb{N}$ s.t. $a_n < \Delta \ \forall n \geq N$
  (a_n) diverges to $-\infty$
- Divergent not implies diverges to ∞. e.g. $\langle 1, -1, 1, -1, \ldots \rangle$ is divergent but not diverges to ∞
- A sequence which is not convergent to any finite point is divergent sequence

I.P.

2.3.7)(a) $\lim \sqrt{n} = \infty$

$\Delta > 0$ be given

GOAL: $\sqrt{n} > \Delta$ i.e. $n > \Delta^2$

Choose N any natural no. greater than Δ^2.

(b) It doesn't diverge to ∞

- (a_n): sequence, $A \subseteq \mathbb{R}$
  There exists an $N \in \mathbb{N}$ s.t. $a_n \in A \ \forall n \geq N$
  "(a_n) eventually enters in A"

- Any $N \in \mathbb{N}$, there exists an $M \in \mathbb{N}$, $M \geq N$ such that $a_M \in A$.
  "(a_n) frequently enters in A".

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$$\frac{|x_n - x|}{\underbrace{\sqrt{x_n}+\sqrt{x}}_{\text{if } <1}} \nleq \frac{|x_n - x|}{\underbrace{1}_{\geq 1}}$$

2.2.8) (a) Frequently

★ $P \Rightarrow Q$ but $Q \nRightarrow P$ or $Q \nRightarrow P$

$\underset{\text{stronger}}{P}$ $\underset{\text{weaker}}{Q}$

(b) Eventually $\Rightarrow$ Frequently

(c) $a_n \in (l-\varepsilon, l+\varepsilon) \to A$ $\quad l-\varepsilon \quad l \quad l+\varepsilon$

$\therefore$ eventually in $V_\varepsilon(l)$.

(d) $a_n \in (1.9, 2.1)$ for infinitely many values of n.

Consider $(2, -2, 2, -2, \ldots)$

~~$a_n \in (1.9, 2.1)$ for all odd values of n~~ or $a_n = 2$ whenever n is odd

$\therefore (a_n)$ isn't eventually but frequently in $(1.9, 2.1)$.

Given: $N \in \mathbb{N}$

Let if possible, $a_n \notin (1.9, 2.1)$ if $n \geq N$.

$a_n \in (1.9, 2.1)$ for at most $N-1$ values of n. $\rightarrow\!\times$

which is a contradiction to given hypothesis.

Exercise 2.3

Courtesy: Stephen Abbott

Exercise 2.3.1. Show that the constant sequence $(a, a, a, a, \dots)$ converges to a.

Exercise 2.3.2. Let $x_n \geq 0$ for all $n \in \mathbf{N}$.
(a) If $(x_n) \to 0$, show that $(\sqrt{x_n}) \to 0$.
(b) If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$.

Exercise 2.3.3 (Squeeze Theorem). Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbf{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Exercise 2.3.4. Show that limits, if they exist, must be unique. In other words, assume $\lim a_n = l_1$ and $\lim a_n = l_2$, and prove that $l_1 = l_2$.

Exercise 2.3.5. Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \dots, x_n, y_n, \dots)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

Exercise 2.3.6. (a) Show that if $(b_n) \to b$, then the sequence of absolute values $|b_n|$ converges to $|b|$.
(b) Is the converse of part (a) true? If we know that $|b_n| \to |b|$, can we deduce that $(b_n) \to b$?

Exercise 2.3.7. (a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

(b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b?
(c) Use (a) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

Exercise 2.3.8. Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):
(a) sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges;
(b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges;
(c) a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges;
(d) an unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded;
(e) two sequences (a_n) and (b_n), where $(a_n b_n)$ and (a_n) converge but (b_n) does not.

Exercise 2.3.9. Does Theorem 2.3.4 remain true if all of the inequalities are assumed to be strict? If we assume, for instance, that a convergent sequence (x_n) satisfies $x_n > 0$ for all $n \in \mathbf{N}$, what may we conclude about the limit?

Exercise 2.3.10. If $(a_n) \to 0$ and $|b_n - b| \leq a_n$, then show that $(b_n) \to b$.

Exercise-2.3

2.3.2) (a) $\varepsilon > 0$ be given

GOAL: $|\sqrt{x_n} - 0| < \varepsilon$ i.e. $x_n < \varepsilon^2$ i.e. $|x_n - 0| < \varepsilon^2$

$(x_n) \to 0$, $\therefore \exists$ an $N \in \mathbb{N} \ni |x_n - 0| < \varepsilon^2 \ \forall n \geq N$

(b) given: $(x_n) \to x$

$\varepsilon > 0$ be given

GOAL: $|\sqrt{x_n} - \sqrt{x}| < \varepsilon$

i.e. $\dfrac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} < \varepsilon$

Also,

$$\frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}}$$, if $x \neq 0$ ($x = 0$, we have done in (a))

Since $(x_n) \to x$, $\exists$ an $N \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon \sqrt{x} \ \forall n \geq N$

2.3.5) $(x_n), (y_n)$

$(z_n) = (x_1, y_1, x_2, y_2, \ldots)$

T.P: (z_n) is cgt $\Leftrightarrow$ (x_n) & (y_n) are both cgt with $\lim x_n = \lim y_n$

($\Rightarrow$) Let (z_n) be convergent, $\therefore \exists$ some $N \in \mathbb{N}$ s.t. $|z_n - l| < \varepsilon \ \forall n \geq N$ — ①
$(z_n) \longrightarrow l$

Claim: $\lim x_n = l$

Given: $\varepsilon > 0$

GOAL: $|x_n - l| < \varepsilon$

$x_1 = z_1,\ x_2 = z_3,\ x_3 = z_5, \ldots, x_{\frac{n+1}{2}} = z_n$, n is odd.

Suppose $M, M+2, M+4, \ldots \longrightarrow$ odd nos. $\geq N$ — ②

$$①\ \&\ ② \Rightarrow \left.\begin{array}{l} |z_M - l| < \varepsilon \Rightarrow |x_{\frac{M+1}{2}} - l| < \varepsilon \\ |z_{M+2} - l| < \varepsilon \Rightarrow |x_{\frac{M+3}{2}} - l| < \varepsilon \\ |z_{M+4} - l| < \varepsilon \Rightarrow |x_{\frac{M+5}{2}} - l| < \varepsilon \\ \vdots \end{array}\right\} \Rightarrow |x_n - l| < \varepsilon \ \forall n \geq \frac{M+1}{2}$$

$\therefore (x_n)$ converges to l.

Similarly, (y_n) converges to l.

($\Leftarrow$) If even entries and odd entries converges to l, then whole sequence converges to l.

2.3.6) (a) $(b_n) \to b$

T.S: $|b_n| \to |b|$

$\varepsilon > 0$ be given.

GOAL: $||b_n| - |b|| < \varepsilon$

$||b_n| - |b|| \leq |b_n - b| < \varepsilon$

2.3.3) Squeeze Theorem or Sandwich Theorem: $(x_n), (y_n), (z_n)$: sequences

$$\underset{\downarrow l}{x_n} \leq y_n \leq \underset{\downarrow l}{z_n} \ \forall n \in \mathbb{N}$$

T.S: $(y_n) \to l$

Proof: $\varepsilon > 0$ be given

There exists an N_1 s.t. $|x_n - l| < \varepsilon \ \forall n \geq N_1$

& There exists an N_2 s.t. $|z_n - l| < \varepsilon \ \forall n \geq N_2$

$N = \max\{N_1, N_2\}$

$$\left.\begin{array}{l} |x_n - l| < \varepsilon \\ |z_n - l| < \varepsilon \end{array}\right] \forall n \geq N \quad \text{i.e.} \quad \begin{array}{l} l - \varepsilon < x_n < l + \varepsilon \\ l - \varepsilon < z_n < l + \varepsilon \end{array} \ \forall n \geq N$$

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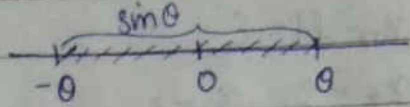
$|\sin\theta| \le |\theta| \ \forall \ \theta \in \mathbb{R}$

$l-\varepsilon < x_n \le y_n \le z_n < l+\varepsilon \ \forall n \ge N$

i.e. $l-\varepsilon < y_n < l+\varepsilon \ \forall n \ge N$

i.e. $|y_n - l| < \varepsilon \ \forall n \ge N.$

Q- Show that $\sin\frac{1}{n} = 0$



Solⁿ: In particular,

let $\theta = \frac{1}{n}$

$|\sin\theta| \le |\theta| \ \forall \ \theta \in \mathbb{R}$

$\Rightarrow -\theta \le \sin\theta \le \theta \ \forall \ \theta \in \mathbb{R}$

$\underset{\to 0}{-\frac{1}{n}} \le \sin\frac{1}{n} \le \underset{\to 0}{\frac{1}{n}} \ \forall n \in \mathbb{N}$

By Squeeze Principle, $\lim \sin\frac{1}{n} = 0$

Q- Evaluate:

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}\right]$$

Solⁿ: $\frac{1}{\sqrt{n^2+n}} + \frac{1}{\sqrt{n^2+n}} + \dots + \frac{1}{\sqrt{n^2+n}} < \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}} < \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+1}} + \dots + \frac{1}{\sqrt{n^2+1}}$

$$\frac{n}{\sqrt{n^2+n}} < a_n < \frac{n}{\sqrt{n^2+1}} \ \forall n \in \mathbb{N}$$

$$\lim \frac{n}{\sqrt{n^2+n}} = \lim \frac{1}{\sqrt{1+1/n}} = \frac{1}{\sqrt{1+\lim\frac{1}{n}}} = \frac{1}{\sqrt{1+0}} = 1$$

$$\lim \frac{n}{\sqrt{n^2+1}} = \lim \frac{1}{\sqrt{1+1/n^2}} = \lim \frac{1}{\sqrt{1+0}} = 1$$

⊛ Result: f: continuous at a.

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

⊛ Limit function commutes with continuous functions.

- Limit as a Sum: $\displaystyle \lim_{n\to\infty}\frac{1}{n}\sum_{r=\phi(n)}^{\psi(n)} f\left(\frac{r}{n}\right) = \int_{\lim_{n\to\infty}\frac{\phi(n)}{n}}^{\lim_{n\to\infty}\frac{\psi(n)}{n}} f(x)\,dx$

Q- ① $\displaystyle \lim_{n\to\infty}\left\{\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}\right\}$ ② $\displaystyle \lim_{n\to\infty}\left\{\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \dots + \frac{n}{n^2+n^2}\right\}$ ($\frac{n}{n^2+n^2} = \frac{1}{2n}$)

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③ $\lim_{n\to\infty} \frac{(n!)^{1/n}}{n}$

④ $\lim_{n\to\infty} \left\{ \frac{1}{n^2} \sec^2 \frac{1}{n^2} + \frac{2}{n^2} \sec^2 \frac{4}{n^2} + \dots + \frac{1}{n} \sec^2 1 \right\}$

Solⁿ: ① $\lim_{n\to\infty} \frac{1}{n} \left\{ \frac{n}{n+1} + \frac{n}{n+2} + \dots + \frac{n}{n+n} \right\} = \lim_{n\to\infty} \frac{1}{n} \left\{ \frac{1}{1+1/n} + \frac{1}{1+2/n} + \dots + \frac{1}{1+\frac{n}{n}} \right\}$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \underbrace{\frac{1}{1+\frac{r}{n}}}_{f\left(\frac{r}{n}\right)}$, $\phi(n) = 1, \psi(n) = n$

$= \int_{\lim_{n\to\infty} \frac{1}{n}}^{\lim_{n\to\infty} \frac{n}{n}} \frac{1}{1+x}\, dx = \left| \ln(1+x) \right|_0^1 = \ln 2 - \ln 1 = \ln 2.$

② $\lim_{n\to\infty} \frac{1}{n} \left\{ \frac{n^2}{n^2+1^2} + \frac{n^2}{n^2+2^2} + \dots + \frac{n^2}{n^2+n^2} \right\}$

$= \lim_{n\to\infty} \frac{1}{n} \left\{ \frac{1}{1+\frac{1^2}{n^2}} + \frac{1}{1+\frac{2^2}{n^2}} + \dots + \frac{1}{1+\frac{n^2}{n^2}} \right\}$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{1+\left(\frac{r}{n}\right)^2} = \int_0^1 \frac{1}{1+x^2}\, dx = \left[\tan^{-1}(x)\right]_0^1 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

④ $\lim_{n\to\infty} \frac{1}{n} \left\{ \frac{1}{n} \sec^2\left(\frac{1}{n}\right)^2 + \frac{2}{n} \sec^2\left(\frac{2}{n}\right)^2 + \dots + \left(\frac{n}{n}\right) \sec^2\left(\frac{n}{n}\right)^2 \right\}$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{r}{n} \sec^2\left(\frac{r}{n}\right)^2 = \int_0^1 x \sec^2(x^2)\, dx = \int_0^1 \frac{1}{2} \sec^2(t)\, dt$

$= \frac{1}{2} \left| \tan(t) \right|_0^1 = \frac{1}{2}\left[\tan(1) - \tan(0)\right] = \frac{\tan(1)}{2} = 0.7787$

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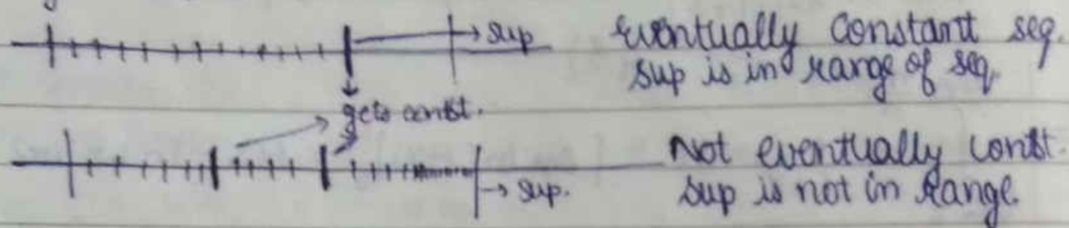
27/8/16

$\sup(a_n) \in \text{Range}(a_n) \to$ eventually const. seq.
$\sup(a_n) \notin \text{Range}(a_n) \to$ Not an eventually const. seq.

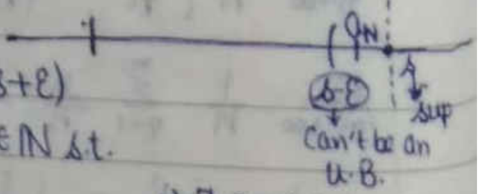
★ Cgt $\Rightarrow$ bdd but bdd $\not\Rightarrow$ cgt
e.g. $(1, -1, 1, -1, \dots)$

- <u>Monotone increasing</u>: If $m_1 > m_2$, then $a_{m_1} \geq a_{m_2}$
  e.g. $\langle 1, 1, 1, \dots \rangle$



This is called <u>Intution</u>
We get cgt. seq. in both the cases.
<u>Monotone Convergence theorem</u>:-
⊛ <u>Result</u>: A monotone bounded sequence converges.

<u>Proof</u>: Without loss of generality, we can take
(a_n): monotone increasing & bounded sequence
By AoC, the supremum of $\{a_n : n \in \mathbb{N}\}$ exists
Let $s = \sup\{a_n : n \in \mathbb{N}\}$
<u>Claim</u>: seq. converges to s $[(a_n) \to s]$
Let $\varepsilon > 0$ be given.



$\Rightarrow \exists$ some a_N s.t. $s-\varepsilon < a_N$

<u>WISH</u>: $|a_n - s| < \varepsilon$ i.e. $a_n \in (s-\varepsilon, s+\varepsilon)$
$s-\varepsilon$ can't be an U.B., so $\exists$ some $N \in \mathbb{N}$ s.t.
$s - \varepsilon < a_N$
M.I. $\Rightarrow$ $s - \varepsilon < a_n < s \;\; \forall n \geq N$
$\Rightarrow |a_n - s| < \varepsilon \;\; \forall n \geq N$

- <u>Series</u>:
  (a_n): sequence

$$\sum_{n\in\mathbb{N}} a_n = \underbrace{a_1 + a_2 + a_3 + a_4 + \dots}_{\text{series}}$$

sequence of partial sums
$$\left\{\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \end{aligned}\right\} \text{Partial sums of } \sum a_n$$

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Achilles
Styx (Magical river)
Achillesheel (Weak pt. of a person)

Telescopic Sum → Only first & last term gets remain

If $s_n \to B$, then we write $\Sigma a_n \to B$ → Sum
"Σa_n converges"

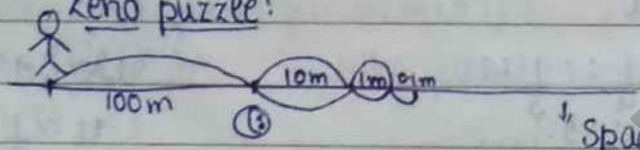
- Geometric series: $a + ar + ar^2 + ar^3 + \dots$

$$S = a + \cancel{ar} + \cancel{ar^2} + \cancel{\dots} + \cancel{ar^{n-1}}$$
$$\underline{-\; Sr = \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \dots + ar^n}$$
$$S(1-r) = a - ar^n \Rightarrow S = \frac{a(1-r^n)}{1-r}$$

$$\lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}, \text{ if } |r|<1$$

$\therefore \Sigma ar^n = \dfrac{a}{1-r}$, if $|r|<1$

e.g. Zeno puzzle:



$100 + 10 + 1 + 0.1 + \dots \cong 111.11$

"Space (track) → Can't be divided infinitely
Mistake in this puzzle

Q: $\Sigma \frac{1}{n^2}$ → Check its convergence.

Sequence of partial sums → Monotone increasing sequence

$$\begin{cases} s_1 = \frac{1}{1^2} \\ s_2 = \frac{1}{1^2} + \frac{1}{2^2} \\ s_3 = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \\ \vdots \end{cases}$$

$$s_m = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{m^2} \le \frac{1}{1} + \frac{1}{2\cdot1} + \frac{1}{3\cdot2} + \dots + \frac{1}{m(m-1)}$$

i.e. $s_m < \frac{1}{1} + \frac{2-1}{2\cdot1} + \frac{3-2}{3\cdot2} + \dots + \frac{m-(m-1)}{m(m-1)}$

i.e. $s_m < 1 + \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \cancel{\dots} + \left(\cancel{\tfrac{1}{m-1}} - \frac{1}{m}\right)$ → Telescopic sum

i.e. $s_m < 2 - \frac{1}{m} < 2$

$s_m < 2 \;\forall\, m \in \mathbb{N} \Rightarrow (s_m)$ is bounded above

By MCT, (s_m) converges

$\therefore \Sigma \frac{1}{n^2}$ converges.

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$\sum \frac{1}{n^2} = \frac{\pi^2}{6}$

★ $\sum \frac{1}{n}$ → Harmonic series

$S_m = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m}$

$$S_{2^k} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^{k-1}} + \dots + \frac{1}{2^k}\right) > 1 + \frac{1}{2}\cdot k$$

$$\frac{1}{4} + \frac{1}{4} \qquad \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \qquad \frac{1}{2^k} + \frac{1}{2^k} \dots \frac{1}{2^k}$$

No of terms in last group: $2^k - (2^{k-1}+1) + 1 = 2^k - 2^{k-1} = 2^{k-1}(2-1) = 2^{k-1}$

S_m is not bounded above, so, it is not bounded and hence, it is not convergent.

$\therefore \sum \frac{1}{n}$ is divergent

★ Ratio of λ ← $1 :: \frac{1}{2} :: \frac{1}{3} :: \frac{1}{4} :: \frac{1}{5} :: \dots$

Ratio of u ← $1 :: 2 :: 3 :: 4 :: 5 :: \dots$

$u\lambda = c \Rightarrow u = \frac{c}{\lambda}$

$u \propto \frac{1}{\lambda}$

★ (a_n): Sequence

$(a_1, a_2, a_3, \dots)$

$n_1 < n_2 < n_3 < n_4 < \dots$ → Sequence of natural nos & is strictly incr

$(a_{n_k})_{k=1}^{\infty}$ ← $(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, \dots)$ → Subsequence of (a_n)

- Definition: Let (a_n) be a sequence and $n_1 < n_2 < n_3 < \dots$ be a strictly increasing sequence of natural nos, then $(a_{n_1}, a_{n_2}, \dots)$ is called subsequence of (a_n)

⊛ Subsequence is not a unique

Eg $(2, 4, 6, 8, \dots)$ Sq

$(4, 8, 10, \dots)$ ✓ Subsequence

$(8, 4, 10, \dots)$ ✗

★ If (a_n): convergent sequence & $(a_n) \to l$, then $(a_{n_k}) \to l$

⊛ Result: Subsequence of a convergent sequence converge at the same point.

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$\boxed{n_k \geq K}$

Proof: Given: $(a_n) \to l$

T.S: $(a_{n_k}) \to l$

Let $\varepsilon > 0$ be given

$\exists$ an $N \in \mathbb{N}$ s.t. $|a_n - l| < \varepsilon \ \forall n \geq N$

$\Rightarrow |a_{n_k} - l| < \varepsilon \ \forall n_k \geq N$

$\therefore (a_{n_k}) \to l$

Q. Show that $\lim b^n = 0$, if $0 < b < 1$.

Sol^n: $(b, b^2, b^3, \ldots) \to$ Monotone decreasing & Bounded below $\Rightarrow$ cgt

$\ldots < b^4 < b^3 < b^2 < b < 1$

By MCT, (b^n) is convergent $\to$ (b^n) would converge to its infimum

Suppose $(b^n) \to l$ (l: infimum of (b^n))

$(b^{2n}) = (b^2, b^4, b^6, \ldots)$: subsequence

$(b^{2n}) \to l$

$\lim b^{2n} = l$

$\Rightarrow \lim (b^n \cdot b^n) = l \Rightarrow (\lim b^n) \cdot (\lim b^n) = l$ (By Algebraic Limit Theorem)

i.e. $l^2 = l \Rightarrow l = 0$ or 1

As infimum $\neq 1$, so, 1 is rejected. $\Rightarrow l = 0$

$\therefore \lim b^n = 0$

$\star$ $\left(\frac{1}{5}, \frac{-1}{5}, \frac{1}{5}, \frac{-1}{5}, \ldots\right) \to$ Is it cgt.? No.

$\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \ldots\right) \to \frac{1}{5}$ & $\left(\frac{-1}{5}, \frac{-1}{5}, \frac{-1}{5}, \ldots\right) \to \frac{-1}{5}$

It cannot cgt. as its subsequences converge at distinct points

- **Cauchy Sequences:** Let $\varepsilon > 0$ be given

  (a_n): sequence

  If $\exists$ some $N \in \mathbb{N}$ such that $|a_m - a_n| < \varepsilon \ \forall m, n \geq N$

  Distance b/w the terms of the seq.

⊛ **Result:** Cauchy sequences are bounded.

T.S: $|a_n| \leq M$

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For min, we use $|a_N| < |a_n| + 1$ i.e. $|a_n| > |a_N| - 1 \; \forall n \geq N$ (M^c)

ε is arbitrary small, so, $2\varepsilon, \ldots, 1000\varepsilon \ldots$ are arbitrary small

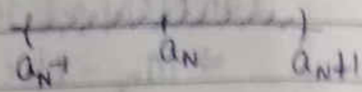
Proof: (a_n): Cauchy sequence

Set $\varepsilon = 1$

$\exists$ an $N \in \mathbb{N}$ s.t. $|a_m - a_n| < \varepsilon \; \forall m, n \geq N$

In particular, $|a_N - a_n| < 1 \; \forall n \geq N$

i.e., $a_n \in (a_N - 1, a_N + 1) \; \forall n \geq N$



As $\big||a_N| - |a_n|\big| < |a_N - a_n| \Rightarrow |a_N| - |a_n| < |a_N - a_n|$ or $|a_n| - |a_N| < |a_N - a_n|$

$\Rightarrow |a_N| - |a_n| < 1$ & $|a_n| - |a_N| < 1 \; \forall n \geq N$

$\Rightarrow$ $\left.\begin{array}{l} |a_N| < |a_n| + 1 \\ |a_n| < |a_N| + 1 \end{array}\right\} \forall n \geq N$ — we use this

Let $M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a_N| + 1\}$

$\Rightarrow |a_n| \leq M \; \forall n$

Result: Cauchy in $\mathbb{R} \Leftrightarrow$ Cauchy Cgt.

Proof: ($\Leftarrow$) Let (a_n) be cgt.

T.S: (a_n) is Cauchy

Suppose $(a_n) \to l$

Let $\varepsilon > 0$ be given, then $\exists$ $N \in \mathbb{N}$ s.t. $|a_n - l| < \frac{\varepsilon}{2} \; \forall n \geq N$

$\Rightarrow |a_m - l| < \frac{\varepsilon}{2} \; \forall m \geq N$

$|a_m - a_n| = |a_m - l + l - a_n| \leq |a_m - l| + |a_n - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall m, n \geq N$

Or $|a_n - l| < \varepsilon$ & $|a_m - l| < \varepsilon \Rightarrow |a_m - a_n| < 2\varepsilon \; \forall m, n \geq N$

($\Rightarrow$) Let (a_n) be Cauchy

T.S: (a_n) is cgt.

(a_n): Cauchy $\Rightarrow$ bounded sequence

$\Rightarrow (a_n)$ has some convergent subsequence, say (a_{n_k}) (By B-W)

Suppose $(a_{n_k}) \to x$

Claim: $(a_n) \to x$

As (a_n) is Cauchy & $\varepsilon > 0$ is given, so, $\exists$ an $N \in \mathbb{N}$ s.t. $|a_m - a_n| < \varepsilon \; \forall n, m \geq N$

As $(a_{n_k}) \to x$, $\exists$ an $M \in \mathbb{N}$ s.t. $|a_{n_k} - x| < \varepsilon \; \forall n_k \geq M$

Let $S = \max\{M, N\}$

$|a_n - x| = |a_n - a_m + a_m - a_{n_k} + a_{n_k} - x|$

$\leq |a_n - a_m| + |a_m - a_{n_k}| + |a_{n_k} - x| < \varepsilon + \varepsilon + \varepsilon$, if $m, n, n_k \geq S$

$\therefore |a_n - x| < 3\varepsilon \; \forall n \geq S$

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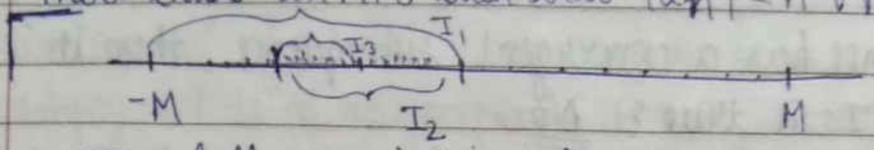
$\exists\, n_2$ s.t. $n_2 > n_1$ as if not then I_2 has finitely many terms *

28/8/16 Imp

- **Bolzano-Weierstrass Theorem:** A bounded sequence has a convergent subsequence.

Proof: (a_n): Bounded sequence
There exists an $M>0$ such that $|a_n| < M \;\forall n \in \mathbb{N}$



I_1: half containing infinitely many terms.
closed interval.
I_2: half of I_1 containing infinitely many terms
$I_1 \cap I_2 \cap I_3 \cap \dots$ $\rightarrow$ all are closed intervals
Nest $\{ I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$
By Nested Interval Property, $\bigcap_{n\in\mathbb{N}} I_n \neq \phi$
$\Rightarrow \exists$ some $x \in \mathbb{R}$ s.t. $x \in \bigcap_{n\in\mathbb{N}} I_n$
Pick a_{n_1} from I_1 and Pick a_{n_2} from I_2 s.t. $n_2 > n_1$
Pick a_{n_3} from I_3 s.t. $n_3 > n_2$
⋮

So, we have got a subsequence (a_{n_k}) s.t. $a_{n_k} \in I_k$
$a_{n_1} \in I_1$ & $x \in I_1$ and maximum distance between x & a_{n_1} is the length of I_1 and so on.
$\therefore$ Maximum distance b/w x & a_{n_k} is the length of I_k and which is becoming smaller & smaller, so, the terms of the subsequence getting closer and closer to x.

Claim: $(a_{n_k}) \rightarrow x$
Given: $\varepsilon > 0$
WISH: $|a_{n_k} - x| < \varepsilon$

$$a_{n_k}, x \in I_k \Rightarrow |a_{n_k} - x| \leq l(I_k) = \frac{M}{2^{k-1}}$$

$$\left(\because l(I_1) = M \rightarrow l(I_2) = \frac{M}{2} \rightarrow l(I_3) = \frac{M}{2^2} \;\dots\; l(I_k) = \frac{M}{2^{k-1}} \right)$$

If we show $\frac{M}{2^{k-1}} < \varepsilon$, we are done.

$$\frac{M}{2^{k-1}} < \varepsilon \text{ i.e. } 2^{k-1} > \frac{M}{\varepsilon} \overset{\text{i.e.}}{\Rightarrow} (k-1)\log_2 2 > \log_2 \frac{M}{\varepsilon} \quad \left(M, \varepsilon > 0 \Rightarrow M/\varepsilon > 0 \right)$$

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$\log_b a = \frac{\log a}{\log b}$ $\lg x = \log_2 x$

i.e. $K > 1 + \log_2 \frac{M}{\varepsilon}$

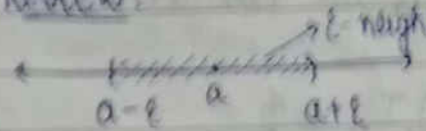
Choose any natural number $N > 1 + \log_2 \frac{M}{\varepsilon}$

$|a_{n_K} - x| < \varepsilon \;\; \forall n_K \geq K \geq N$

* If a sequence has a convergent subsequence, then it is bounded? Is it true? No

e.g: $\langle 1, 2, 3, 1, 4, 1, 5, \ldots \rangle$ is not bounded but has a convergent subsequence $\langle 1, 1, 1, 1, \ldots \rangle$

- Review:



$V_\varepsilon(a) := \{x \in \mathbb{R} : |x - a| < \varepsilon\}$

$A \neq \phi$, $A \subseteq \mathbb{R}$ $\quad V_\varepsilon(a) \subseteq A$, a is interior

Open set: each point is "interior"

- Definition: A set $O \subseteq \mathbb{R}$ is said to be open if for all $a \in O$, $\exists$ $V_\varepsilon(a)$ s.t. $V_\varepsilon(a) \subseteq O$

* ε depends on 'a' as when the pt 'a' comes closer to the boundary the radius (ε) decreases

Q Which of the following are open?

① $[a, b)$ ② (a, b) ③ $\mathbb{Q}$

④ $\mathbb{R}$ ⑤ $\mathbb{N}$ ⑥ The empty set ϕ

⑦ A non empty finite set

Solⁿ: ①

a is not interior

$\nexists$ any $\varepsilon > 0$ s.t. $V_\varepsilon(a) \subseteq [a, b)$

② $\alpha \in (a, b)$

$\varepsilon = \min\{|\alpha - a|, |\alpha - b|\}$

$V_\varepsilon(\alpha) \subseteq (a, b)$

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Card $V_\varepsilon(a) = c$

(*) Every open interval is an open set.

(*) A nonempty set with cardinality $< c$ (cardinality of $\mathbb{R}$) can't be open

(3) Cardinality of $\mathbb{Q} < c$, $\therefore$ $\mathbb{Q}$ can't be open

(5) Cardinality of $\mathbb{N} < c$, $\therefore$ $\mathbb{N}$ can't be open.

(6) Empty set is always open. ($P \Rightarrow Q$, $P \to$ False, then $Q \to$ true)
$a \in \emptyset \Rightarrow \emptyset \subseteq \mathbb{R}$

* $\{O_\lambda : \lambda \in \Lambda\}$ $\longrightarrow$ Collection of O_λ's, where $\lambda \in \Lambda$
indexing set

$O = \bigcup_{\lambda \in \Lambda} O_\lambda$ is it open?
arbitrary union

Let $a \in O$ i.e. $a \in \bigcup_{\lambda \in \Lambda} O_\lambda$

Then $\exists$ some O_{λ_0} s.t. $a \in O_{\lambda_0} \longrightarrow$ open

There exists $V_\varepsilon(a)$ s.t. $V_\varepsilon(a) \subseteq O_{\lambda_0}$
But $O_{\lambda_0} \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$
$\Big\} \Rightarrow V_\varepsilon(a) \subseteq O$ (ε-ball)

(*) Result: (1) The union of any number (finite / infinite) of open sets is open.

(2) $\{O_{\lambda_i} : i = 1, 2, \dots, n\}$: Finite collection of opensets.
The intersection of a finite number of open sets is open.

Proof: T.S: $O = \bigcap_{i=1}^{n} O_{\lambda_i}$ is open

Let $a \in \bigcap_{i=1}^{n} O_{\lambda_i}$

$a \in O_{\lambda_i}$ $\forall i = 1, 2, \dots, n$

$a \in O_{\lambda_1} \Rightarrow \exists$ an $V_{\varepsilon_1}(a)$ s.t. $V_{\varepsilon_1}(a) \subseteq O_{\lambda_1}$

$a \in O_{\lambda_2} \Rightarrow \exists$ an $V_{\varepsilon_2}(a)$ s.t. $V_{\varepsilon_2}(a) \subseteq O_{\lambda_2}$

$\vdots$

$a \in O_{\lambda_n} \Rightarrow \exists$ an $V_{\varepsilon_n}(a)$ s.t $V_{\varepsilon_n}(a) \subseteq O_{\lambda_n}$

$\varepsilon := \min\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$

$\therefore V_\varepsilon(a) \subseteq \bigcap_{i=1}^{n} O_{\lambda_i}$

* If we consider the intersection of infinite number of open sets, then $\min = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \dots\} = 0$ or may not be exist, which creates problem, so, we take intersection of finite number of open sets.

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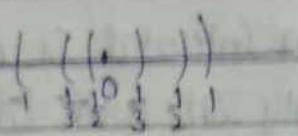
No. of elements of A in $V_\varepsilon(\alpha)$ "$V_\varepsilon(\alpha) \cap A$ has element other than α
or $V_\varepsilon(\alpha)$ intersects with A at some point other than α.

e.g. $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$ ← closed set



⑥ Intersection of open sets may not be open

3/9/16

*① $(a-\varepsilon, a+\varepsilon) \not\subseteq \mathbb{Q}$ for all $\varepsilon > 0 \Rightarrow \mathbb{Q}$ is not open
Irrationals are there

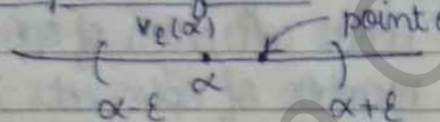
② $a \in \mathbb{N}$
$(a-\varepsilon, a+\varepsilon) \not\subseteq \mathbb{N}\ \forall \varepsilon > 0 \rightarrow \mathbb{N}$ is not open

③ $A = \{a_1, a_2, \dots, a_n\}, n < \infty \rightarrow$ Finite set
$(a_1-\varepsilon, a_1+\varepsilon) \subseteq A$? No, Infinite set can't be subset of finite set
Infinite set   Finite set

④ Empty set (ϕ) is open set
[b/c $a \in A \Rightarrow$ a is an interior pt. of A
$P \Rightarrow Q$, as P is never true so, $P \Rightarrow Q$ is always true]

⑤

- Limit point of a set: $A \subseteq \mathbb{R}, A \neq \phi, \alpha \in \mathbb{R}$



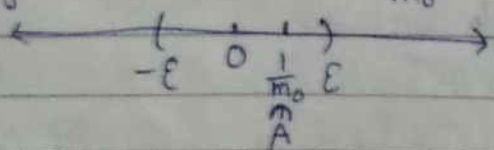
A may be finite or infinite

{Take any nbhd of 'α'
{There exists an element of A, different from α.
→ α is a limit point of A.

⑦ Limit point of a set may not belong to the set.

Q $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Show that 0 is a limit point of A.

Solⁿ: Any $\varepsilon > 0$, $\exists\ m_0 \in \mathbb{N}$ s.t. $\frac{1}{m_0} < \varepsilon$ (By Archimedian Property)



- Definition: Let $\phi \neq A \subseteq \mathbb{R}$. Then $\alpha \in \mathbb{R}$ is called a limit point of A if for each $\varepsilon > 0$, $V_\varepsilon(\alpha)$ intersects A at some point other than

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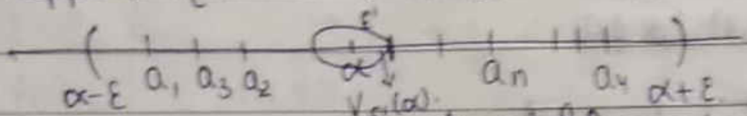
⊛ Result: α is a limit point of $A \Leftrightarrow$ every ε-nbhd of α contains infinitely many points of A.

Proof: ($\Rightarrow$) Let α be a limit point of A and $\varepsilon > 0$ be given.

T.S: $V_\varepsilon(\alpha)$ has an infinite no. of elements of A.

Let if possible, $V_\varepsilon(\alpha) \cap A$ be finite

Suppose $V_\varepsilon(\alpha) \cap A = \{a_1, a_2, \dots, a_n\}$



has no element of $A \Rightarrow \alpha$ is not limit pt. of A ⨳

$\varepsilon' = \min\{|\alpha - a_1|, |\alpha - a_2|, \dots, |\alpha - a_n|\}$

$V_{\varepsilon'}(\alpha)$ has no point of A different from $\alpha \Rightarrow \alpha$ is not a limit point of A. ⨳

[We can't take any $a_i \in V_\varepsilon(\alpha) \cap A$ equal to α, as if we do that then $\varepsilon' = \min\{|\alpha - a_1|, \dots, |\alpha - a_n|\} = 0 \Rightarrow\Leftarrow$ as $\varepsilon' > 0$]

pts. of A going closer to α.

($\Leftarrow$) Trivial.

(as if $V_\varepsilon(\alpha)$ contains infinitely many pts. of A, then infinitely many pts. of A are going closer to $A \Rightarrow \alpha$ is its limit point)

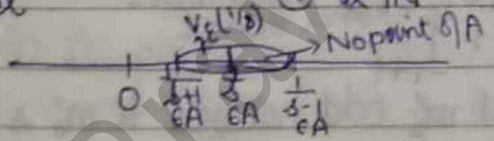
huge accumulation is increasing

Q- Find the limit points.

① $\{\frac{1}{n} : n \in \mathbb{N}\}$ ② $\{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}\}$

③ $\mathbb{Q}$ ④ $\mathbb{N}$ ⑤ $\{1, 2, 3\}$

Sol^n: ①



, $A = \{\frac{1}{n} : n \in \mathbb{N}\}$

$\varepsilon = \min\{\frac{1}{s-1} - \frac{1}{s}, \frac{1}{s} - \frac{1}{s+1}\}$

$V_\varepsilon(\frac{1}{s})$ has no point of $A \Rightarrow \frac{1}{s}$ is not a limit point of A.

$s = 1$

$V_{1/2}(1)$ has no point of A different from 1.

$\Rightarrow 1$ is not a limit point of A.

We don't take $s = 1$ as $\frac{1}{s-1}$ doesn't exist.

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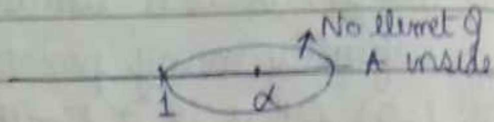
Synonyms ← { Limit point, Accumulation point, Cluster point }

Let $\alpha > 1$

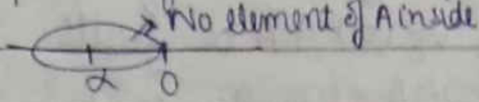
Can α be a limit point of A?



$\varepsilon = \alpha - 1$

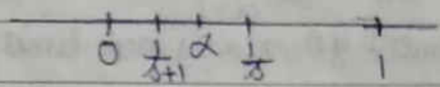
$V_\varepsilon(\alpha)$ has no point of A $\Rightarrow \alpha$ is not a limit point of A

Let $\alpha < 0$



$\varepsilon = -\alpha$

Let $\alpha \in (0,1)$, $\alpha \notin A$



There exists an $s \in \mathbb{N}$ s.t. $\frac{1}{s+1} < \alpha < \frac{1}{s}$

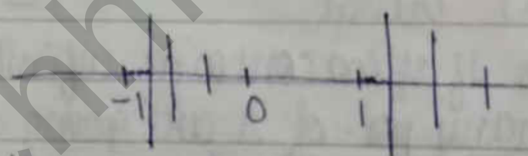
Let $\varepsilon = \min\left\{\alpha - \frac{1}{s+1}, \frac{1}{s} - \alpha\right\}$

$V_\varepsilon(\alpha) \cap A = \phi \Rightarrow \alpha$ is not a limit point of A.

$A' = \{0\}$

Derived set of A: collection of all limit point of A

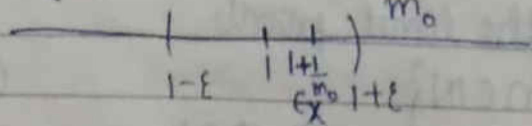
★ $X = \left\{(-1)^n + \frac{1}{n} : n \in \mathbb{N}\right\}$



$X' = \{1, -1\}$ → Limit points of X

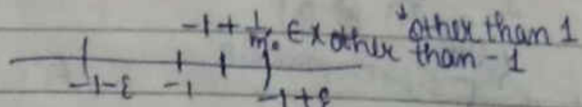
By Archimedean property, $\exists m_0 \in 2\mathbb{N}$ s.t. $\frac{1}{m_0} < \varepsilon$

$\Rightarrow 1 + \frac{1}{m_0} < 1 + \varepsilon$



$\Rightarrow 1 \in X'$

ly, $-1 \in X'$

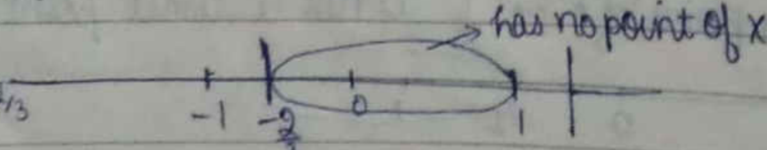


By Archimedean property, $\exists$ odd natural # m_0' s.t.

$\frac{1}{m_0'} < \varepsilon \Rightarrow -1 + \frac{1}{m_0'} < -1 + \varepsilon$

$-1+1=0$

$-1+1/3 = -2/3$



$\Rightarrow 0$ is not a limit point of X.

② $B = \left\{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}\right\}$

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Fix m, vary n $\quad \frac{1}{m_0}-\varepsilon \quad \frac{1}{m_0} \quad \frac{1}{m_0}+\varepsilon \quad \frac{1}{m_0}+\frac{1}{n_0} \in B$ $\qquad \frac{1}{m}+\frac{1}{1},\ \frac{1}{m}+\frac{1}{2},\ \frac{1}{m}+\frac{1}{3},\ \vdots$

$\varepsilon>0$ be given, $\exists\, n_0 \in \mathbb{N} \ni \frac{1}{n_0} < \varepsilon \Rightarrow \frac{1}{m_0}+\frac{1}{n_0} < \frac{1}{m_0}+\varepsilon$

Vary both m and n, $\quad 0 \quad \varepsilon$

$-\varepsilon \quad 0 \quad \varepsilon$, There exists $m_0, n_0 \in \mathbb{N} \ni \frac{1}{m_0} < \frac{\varepsilon}{2}$ & $\frac{1}{n_0} < \frac{\varepsilon}{2}$

$\Rightarrow \frac{1}{m_0}+\frac{1}{n_0} < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon = 0+\varepsilon$

$B' = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$

(3) $\alpha \in \mathbb{R}$, For any $\varepsilon>0$, $(\alpha-\varepsilon, \alpha+\varepsilon)$ contains infinitely many rational numbers. $\quad \alpha-\varepsilon \quad \alpha \quad \alpha+\varepsilon$

$\mathbb{Q}' = \mathbb{R}$

(4) $\mathbb{N}' = \phi$ $\quad$ (at most one pt. of $\mathbb{N}$ other than α) $\quad \frac{1}{2} \quad 1 \quad \frac{3}{2} \quad 2 \quad 3 \quad 4 \quad 5$

$\alpha \in \mathbb{R}$, $(\alpha-\frac{1}{2}, \alpha+\frac{1}{2}) \cap \mathbb{N}$ has at most one element of $\mathbb{N}$

i.e. $|(\alpha-\frac{1}{2}, \alpha+\frac{1}{2}) \cap \mathbb{N}| \le 1$ but we want infinitely many elements $\Rightarrow \alpha$ is not a limit point of $\mathbb{N}$

$\Rightarrow \alpha \notin \mathbb{N}' \Rightarrow \mathbb{N}' = \phi$ (as it has no α i.e. no real no can't be limit point of $\mathbb{N}$)

V. Imp.

(*) **Result:** $\alpha \in A' \Leftrightarrow$ There exists a $\overset{a_n \in A\ \forall\, n \in \mathbb{N}}{\overline{\text{sequence } (a_n) \text{ in } A}}$ such that $a_n \to \alpha$, where $a_n \neq \alpha\ \forall\, n \in \mathbb{N}$ $\quad$ ((a_n) can't be const. seq.)

e.g. $\langle 1,1,2,2,1,1,2,2,\dots \rangle$ is in $\{1,2,3\}$

Proof: $(\Rightarrow)$ Let $\alpha \in A'$

$\alpha-1 \quad \alpha-\frac{1}{2} \quad \alpha-\frac{1}{3} \quad \alpha-\frac{1}{4} \quad \alpha \quad \alpha+\frac{1}{4} \quad a_3 \quad \alpha+\frac{1}{3} \quad a_2 \quad \alpha+\frac{1}{2} \quad a_1 \quad \alpha+1$

$(\alpha-1, \alpha+1)$ has an element of A, other than α, say a_1, $|a_1-\alpha|<1$

$(\alpha-\frac{1}{2}, \alpha+\frac{1}{2})$ has an element of A, other than α, say a_2, $|a_2-\alpha|<\frac{1}{2}$

$(\alpha-\frac{1}{3}, \alpha+\frac{1}{3})$ has an element of A, other than α, say a_3, $|a_3-\alpha|<\frac{1}{3}$

$\vdots$

$(\alpha-\frac{1}{k}, \alpha+\frac{1}{k})$ has an element of A, other than α, say a_k, $|a_k-\alpha|<\frac{1}{k}$

$\vdots$

Claim: $(a_n) \to \alpha$

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Given: $\varepsilon > 0$

$|a_k - \alpha| < \frac{1}{k} \quad \forall k \in \mathbb{N}$ —— (*)

By Archimedean Property, $\exists m_0 \in \mathbb{N}, \ni \frac{1}{m_0} < \varepsilon$

$$\frac{1}{k} \le \frac{1}{m_0} < \varepsilon \quad \forall k \ge m_0 \quad \text{—— } (**)$$

From (*) & (**)

$$|a_k - \alpha| < \varepsilon \quad \forall k \ge m_0$$

★ $\mathbb{Q}' = \mathbb{R}$

$y \in \mathbb{R}$, we can find a sequence of rational numbers which converges to y.

(sequence in $\mathbb{Q}$) $\to \pi$ is $\{3, 3.1, 3.141, 3.1415, \dots\} \to \pi$
decimal expansion of π

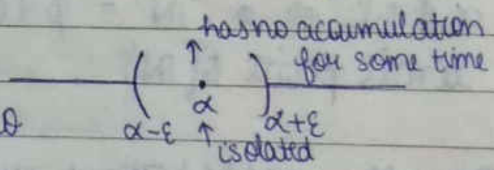
a_n: exact value of π, upto n decimal points.

- <u>Isolated point</u>: $A \subseteq \mathbb{R}, A \ne \phi$

A number $\alpha \in A$ is called an isolated point of A if α is not a limit point of A.

Why the word isolated?

$\alpha \notin A' \Rightarrow \exists V_\varepsilon(\alpha)$ s.t. $V_\varepsilon(\alpha)$ has no element of A other than α.



(*) Isolated point of A $\longrightarrow$ element of A
Limit point of A $\longrightarrow$ may not be an element of A

⑤ $C = \{1, 2, 3\}$ has no limit point but each pt. is isolated pt.

(*) Each element of a finite set is its isolated point.

(*)★ Collection of isolated point of $A = A \setminus A'$

e.g: ① $\mathbb{Q} \setminus \mathbb{R} = \phi \Rightarrow \mathbb{Q}$ has no isolated point

② $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \setminus 0 \Rightarrow$ every point is an isolated point.
$0 \notin A$ as $A \setminus A' = A$ (as $A' = \{0\}$)

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③ $\mathbb{N}\setminus\mathbb{N}' = \mathbb{N}\setminus\phi = \mathbb{N} \Rightarrow$ Each point is an isolated point

- Discrete set: Each element is an isolated point.
  eg: $\{\frac{1}{n}\ n\in\mathbb{N}\}$, $\mathbb{N}$

★ If the limit points belong to the set, i.e., $A'\subseteq A$, then A: closed set
$\alpha\in A'$ & $A'\subseteq A$
$\exists$ a sequence (a_n) in A s.t. $(a_n)\longrightarrow\alpha$, $\alpha\in A$, $a_n\neq\alpha$

- Definition: A set $A\subseteq\mathbb{R}$ is called a closed set, if it contains all its limit points. In Symbols, $A'\subseteq A$.

Q Are the following sets closed?
① $\mathbb{N}$ ② $\mathbb{Q}$ ③ $\mathbb{R}\setminus\mathbb{Q}$ ④ $\{1,2,3,4\}$ ⑤ $\mathbb{R}$ ⑥ $\{\frac{1}{n}:n\in\mathbb{N}\}$

Solⁿ ① $\mathbb{N}'=\phi\subseteq\mathbb{N} \Rightarrow \mathbb{N}$ is closed
② $\mathbb{Q}'=\mathbb{R}\nsubseteq\mathbb{Q} \Rightarrow \mathbb{Q}$ is not closed.
③ $(\mathbb{R}\setminus\mathbb{Q})'=\mathbb{R}\nsubseteq\mathbb{R}\setminus\mathbb{Q} \Rightarrow \mathbb{R}\setminus\mathbb{Q}$ is not closed.
④ $A=\{1,2,3,4\}$ has no limit point. $\Rightarrow A'=\phi\subseteq A \Rightarrow A$ is closed

⊛ If $A'=\phi$, then A is closed.
★ Discrete set is a closed set? No
eg: $\{\frac{1}{n}:n\in\mathbb{N}\}$ is discrete but not closed.
★ $A\subseteq B \Rightarrow A'\subseteq B'$? Yes
Let $\alpha\in A' \Rightarrow V_\varepsilon(\alpha)$ has an element of A, other than α $\forall\varepsilon>0$
$\Rightarrow V_\varepsilon(\alpha)$ has an element of B, other than α $\forall\varepsilon>0$ ($\because A\subseteq B$)
$\Rightarrow \alpha\in B'$

$\therefore A'\subseteq B'$
⊛ $A\subseteq B \Rightarrow A'\subseteq B'$.

⑤ $\mathbb{Q}'=\mathbb{R}$.
$\mathbb{Q}\subseteq\mathbb{R} \Rightarrow \mathbb{Q}'\subseteq\mathbb{R}' \Rightarrow \mathbb{R}\subseteq\mathbb{R}' \Rightarrow \mathbb{R}'=\mathbb{R} \Rightarrow \mathbb{R}$ is closed. (as $\mathbb{R}\subseteq\mathbb{R}$)
Universal set and no set is bigger than it.

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⑥ $A = \{\frac{1}{n} : n \in \mathbb{N}\}$

$A' = \{0\} \not\subseteq A \Rightarrow A$ is not closed

* Universal set (ℝ) is always closed as it has all its limit points inside it.

★ "Sets are not doors"

Doors: Open ⇒ Not closed and Close ⇒ Not Open

But this condition is not in sets.

e.g.: ∅, ℝ : open & closed] clopen sets → closed as well as open

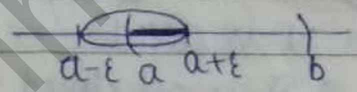
② $(a, b]$ is not open ($\because$ b is not interior)

↓ is not closed ($\because$ a is a limit point, but not an element of the set)

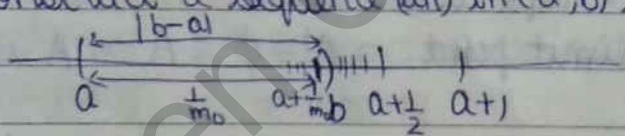
Neither open nor closed

③ (a, b) is open and not closed

⇒ a is its limit point



Q- (a) construct a sequence (a_n) in (a,b) s.t. $(a_n) \to a$, $a_n \neq a$.



$\left(a+1, a+\frac{1}{2}, a+\frac{1}{3}, a+\frac{1}{4}, \ldots\right) \to a$

$\exists m_0 \in \mathbb{N}$ s.t. $\frac{1}{m_0} < |b-a| \Rightarrow a+\frac{1}{m_0} \in (a,b)$

So, the required sequence is $\left(a+\frac{1}{m_0}, a+\frac{1}{m_0+1}, a+\frac{1}{m_0+2}, \ldots\right)$

(b) Search a sequence (b_n) in (a,b) s.t. $(b_n) \to b$, $b_n \neq b$

$\exists m_0' \in \mathbb{N}$ s.t. $\frac{1}{m_0'} < |b-a| \Rightarrow b-\frac{1}{m_0'} \in (a,b)$ $\left(b-\frac{1}{m_0'}, b-\frac{1}{m_0'+1}, b-\frac{1}{m_0'+2}, \ldots\right)$

* Result: $\sigma \subseteq \mathbb{R}$

σ is open $\Leftrightarrow \sigma^c$ is closed.

Proof: (⇒) σ is open

T.S: σ^c is closed

Let if possible, σ^c be not closed

There exists some $\alpha \in \mathbb{R}$ s.t. $\alpha \in (\sigma^c)'$ but $\alpha \notin \sigma^c$

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$\mathcal{O}^c$ is closed $\Rightarrow (\mathcal{O}^c)' \subseteq \mathcal{O}^c$
x $\Leftarrow$ x

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$\Rightarrow \alpha \in \mathcal{O}$
But $\mathcal{O}$ is open, $\exists$ some $\varepsilon > 0$ s.t. $(\alpha-\varepsilon, \alpha+\varepsilon) \subseteq \mathcal{O}$
i.e. $(\alpha-\varepsilon, \alpha+\varepsilon) \cap \mathcal{O}^c = \phi \Rightarrow \alpha \notin (\mathcal{O}^c)'$ i.e. α is not a limit pt. of $\mathcal{O}^c$
($\Leftarrow$) Let $\mathcal{O}^c$ be closed.
T.S: $\mathcal{O}$ is open.
Suppose $a \in \mathcal{O} \Rightarrow a \notin \mathcal{O}^c \Rightarrow a$ is not limit point of $\mathcal{O}^c$ (as $\mathcal{O}^c$ is closed)
$\Rightarrow a \notin (\mathcal{O}^c)'$
$\Rightarrow \exists$ an $\varepsilon > 0$ s.t. $(a-\varepsilon, a+\varepsilon)$ has no element of $\mathcal{O}^c$ other than a
$\Rightarrow (a-\varepsilon, a+\varepsilon) \subseteq \mathcal{O} \Rightarrow \mathcal{O}$ is open

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- F: closed set in $\mathbb{R}$
  $F' \subseteq F$
  $a \in A' \Leftrightarrow$ There exists a seq. (a_n) in A s.t. $(a_n) \to a$, $a_n \neq a$ $\to$ (R1)
  Cauchy in $\mathbb{R}$ (compulsory) $\Leftrightarrow$ Convergent $\to$ (R2)
  Take any cauchy sequence (a_n) in F, $(a_n) \to a \in F$ ($\because$ F is closed)
  limit of $(a_n) \Rightarrow$ limit point of F

(*) **Result:** A set F is closed iff any Cauchy sequence in F, converges to a limit, which is an element of F

Proof: ($\Rightarrow$) Done
($\Leftarrow$) T.S: F is closed, i.e., $F' \subseteq F$
Let $a \in F'$
Then $\exists$ a seq. (x_n) in F s.t. $(x_n) \to a$, $x_n \neq a$. (Using (R1))
HYPOTHESIS: Any cauchy sequence in F, converges to a limit, which is an element of F.
$\Rightarrow (x_n)$ is Cauchy in F s.t. $(x_n) \to a$ (Using (R2))
$\Rightarrow a \in F$ (Using Hypothesis)

Eg: (1) $\mathbb{Q}$
$\{3, 3.1, 3.14, \ldots\}$ is convergent in $\mathbb{R} \Rightarrow$ Cauchy in $\mathbb{R}$.
$\Rightarrow$ Cauchy in $\mathbb{Q}$
$(a_n) \longrightarrow \pi \notin \mathbb{Q}$

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⊛ every closed interval is a closed set.

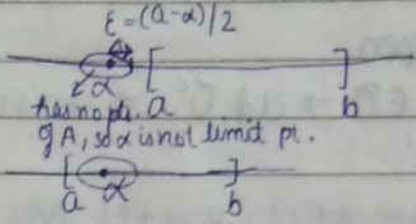
$\therefore \mathbb{Q}$ is not closed as we find a cauchy seq. in $\mathbb{Q}$ which doesn't converge to a limit, which is an element of $\mathbb{Q}$ ($\because \pi \notin \mathbb{Q}$)

② $[a,b] = A$

if $\alpha < a$, $\alpha \in A'$? No

if $\alpha > b$, $\alpha \notin A'$

if $a \le \alpha \le b$, $\alpha \in A'$? Yes



$\therefore A' = [a,b] = A \Rightarrow A' \subseteq A$, $\therefore [a,b]$ is closed.

Or Take a Cauchy sequence (a_n) in $A = [a,b]$

$\Rightarrow$ (a_n) is Cauchy in $\mathbb{R}$

$\Rightarrow$ (a_n) is Convergent

Let $(a_n) \to x$

(a_n) in A, $a \le a_n \le b \ \forall n \in \mathbb{N}$

$\Rightarrow a \le x \le b$ (By Order Limit Theorem)

i.e. $x \in [a,b]$ i.e. $x \in A$

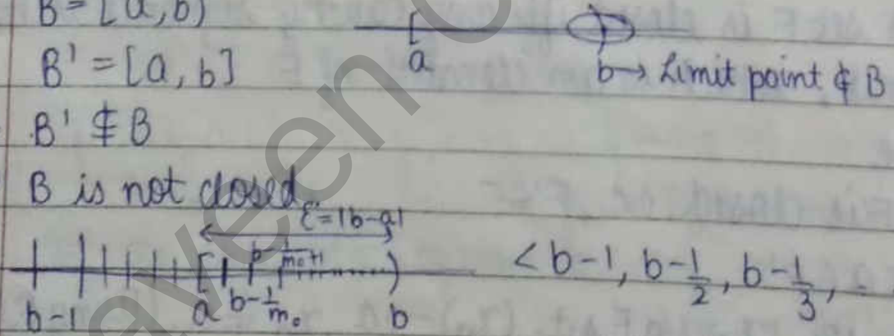
$\therefore [a,b]$ is closed.

③ $B = [a,b)$

$B' = [a,b]$

$\Rightarrow B' \nsubseteq B$

$\Rightarrow$ B is not closed



$\langle b-1, b-\frac{1}{2}, b-\frac{1}{3}, \ldots \rangle$

By Archimedean Property, $\exists\, m_0 \in \mathbb{N} \ni \frac{1}{m_0} < \varepsilon$.

$\Rightarrow \frac{-1}{m_0} > -\varepsilon \Rightarrow b - \frac{1}{m_0} > b - \varepsilon = a$

$(c_n) = \langle \cancel{b - \frac{1}{m_0}}, \overset{c_1}{b - \frac{1}{m_0+1}}, \overset{c_2}{b - \frac{1}{m_0+2}}, \ldots \rangle$ seq. in B

Claim: $(c_n) \to b$

$\varepsilon > 0$ be given

GOAL: $|c_n - b| < \varepsilon$

i.e. $\left| b - \frac{1}{m_0+n} - b \right| < \varepsilon$ i.e. $\frac{1}{m_0+n} < \varepsilon$

⑥ A cauchy sequence in a finite set is always eventually constant.

By Arch. Prop., $\exists$ an $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$

$\Rightarrow \frac{1}{n} < \varepsilon \ \forall n \geq N$ $\left(\cancel{\frac{1}{m_0+N}} \cancel{<} \cancel{\frac{1}{N}} \cancel{< \varepsilon}\right)$

$\Rightarrow \frac{1}{m_0+n} < \frac{1}{n} < \varepsilon \ \forall n \geq N$

$\therefore \frac{1}{m_0+n} < \varepsilon \ \forall n \geq N$

So, (c_n): Cauchy seq in B, which converges to $b \notin B$.

$\Rightarrow$ B is not closed set.

④ $\{1,2,3\} = C$

Let (a_n): Cauchy in C [$\xrightarrow{\text{No use}}$ (a_n): Cauchy in $\mathbb{R}$ $\Rightarrow (a_n)$ is convergent.

Let $(a_n) \to l$

Claim: $l \in C$]

$\varepsilon = 1/2$, there exists $N \in \mathbb{N}$ s.t. $|a_n - a_m| < \frac{1}{2} \ \forall n \geq N, m \geq N$ ($\because$ Cauchy)

In particular, $|a_N - a_{N+1}| < \frac{1}{2}$

$\Rightarrow a_N = a_{N+1}$ ($\because$ minimum distance b/w 2 distinct terms of C is 1)

So, $|a_N - a_m| < \frac{1}{2} \ \forall m \geq N$

$\Rightarrow a_N = a_m \ \forall m \geq N \Rightarrow$ seq. (a_n) is eventually constant.

$(a_n) \to a_N \in C \Rightarrow C$ is closed.

⑤ $D = \{1, 4, 8\}$

Take any $\varepsilon < 3$, to get contradiction (equality)

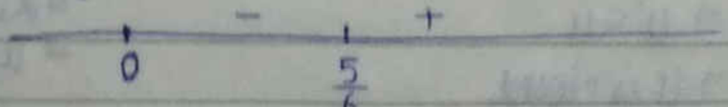
Choose $0 < \varepsilon <$ Minimum distance among the elements of A.

JAN 2014 Let $S = \{x \in \mathbb{R} : x^6 - x^5 \leq 100\}$ ($\to f^{-1}((-\infty, 100])$) & $T = \{x^2 - 2x : x \in (0, \infty)\}$. Then $S \cap T$ is

✓(a) closed and bounded (b) closed but not bounded

(c) bounded but not closed (d) neither closed nor bounded

Solⁿ: $f(x) = x^6 - x^5 \Rightarrow f'(x) = 6x^5 - 5x^4 = x^4(6x-5)$



$f'(0) = 0$

$\lim_{x \to -\infty} x^6 - x^5 = \infty$ ($\because$ power of x is even)

$\lim_{x \to \infty} x^6 - x^5 = \lim_{x \to \infty} x^6 \left(1 - \frac{1}{x}\right) = \infty$

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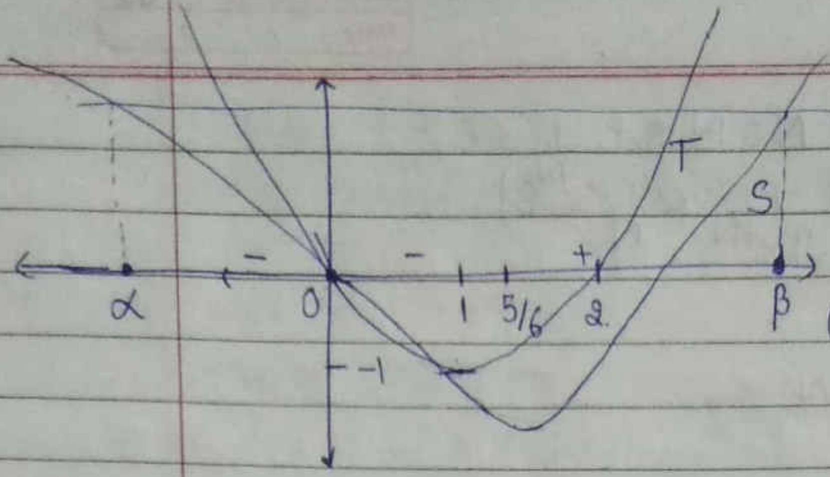


$y=100$ $T=[-1,\infty)$ (−1 ↓ image of 1)

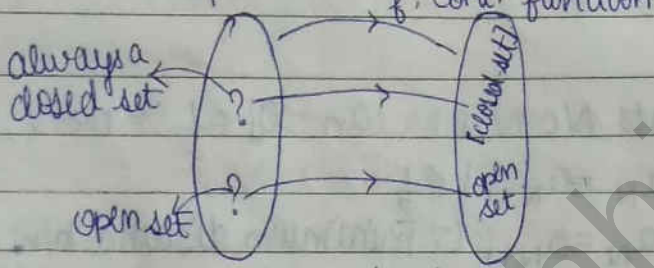
$S=[\alpha,\beta]$

(To find α,β put $x^6-x^5=100$)

S is bounded but & T is not bounded above, so, not bounded

$S\cap T\subseteq S \Rightarrow S\cap T$ is bounded

$S\cap T=[\max \text{ of }(\alpha,-1),\beta] \Rightarrow$ closed & bounded



⊛ The preimage of closed set (open set) under a continuous function is closed (open).

JAM 2008 The set $U=\{x\in\mathbb{R} : \sin x=\frac{1}{2}\}$ is

(a) open (b) closed

(c) open and closed (d) neither open nor closed.

Sol^n: $\sin x=\frac{1}{2} \Rightarrow \sin x=\sin\frac{\pi}{6}$

$\Rightarrow x=n\pi+(-1)^n\frac{\pi}{6}$; $n\in\mathbb{Z}$ $\left(\because \sin x=\sin\theta \Rightarrow x=n\pi+(-1)^n\theta,\ n\in\mathbb{Z}\right.$

$U=\{n\pi+(-1)^n\frac{\pi}{6} : n\in\mathbb{Z}\}$

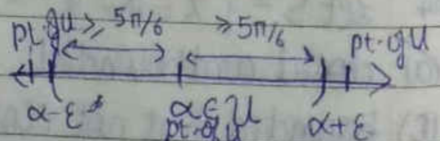
$a\in\mathbb{R}$, $0<\varepsilon<\frac{5\pi}{6}$

$|(a-\varepsilon,a+\varepsilon)\cap U|\leq 1 \Rightarrow a\notin U'$

$\therefore U'=\phi$

$\Rightarrow U'\subseteq U$

$\Rightarrow U$ is closed



$(\alpha-\varepsilon,\alpha+\varepsilon)\nsubseteq U$

$\Rightarrow \alpha$ is not interior pt.

$\Rightarrow U$ is not open.

JAM 2011 Let $E=\left\{\frac{n}{n+1} : n\in\mathbb{N}\right\}$, $F=\left\{\frac{1}{1-x} : 0\leq x<1\right\}$. Then

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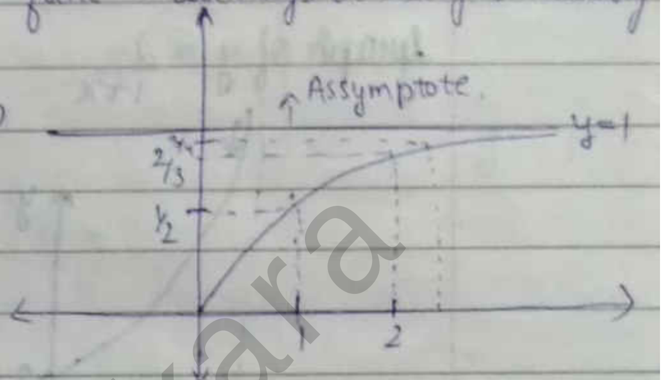
Concave $f''(x)<0$ i.e. $\frac{d}{dx} f' < 0$ (derivative decreasing)

Convex $f''(x)>0$ i.e. $\frac{d}{dx} f' > 0$ (derivative increasing)

① E and F both are closed.
② E is closed but F is not closed.
✓③ F is closed but E is not closed
④ neither E nor F is closed.

Sol^n $f(x) = \frac{x}{x+1}$, $x>0$

$f'(x) = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2} > 0 \Rightarrow$ func^n is always strictly increasing

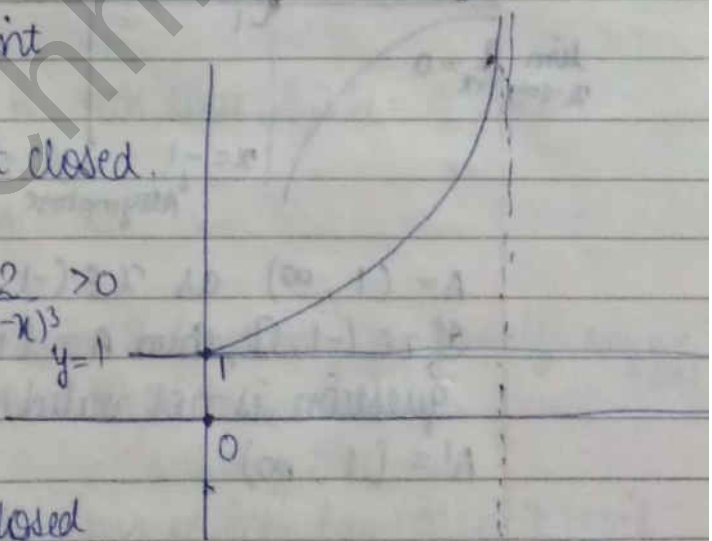


It passes through origin at $x=0$
$f'(0)=1$
$f''(x)<0$

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x+1}$

$= \lim_{x\to\infty} \frac{1}{1+1/x} = 1$

$\therefore$ 1 is the only limit point
Show $1 \in E'$, analytically
$1 \in E'$ but $1 \notin E \Rightarrow E$ is not closed.



$g(x) = \frac{1}{1-x}$ $\quad g'(0)=1$
$g(0)=1$ $\quad g''(x) = \frac{2}{(1-x)^3} > 0$
$g'(x) = \frac{+1}{(1-x)^2} > 0$

$F = [1,\infty)$
Check $F' = [1,\infty) \subseteq F \Rightarrow F$ is closed

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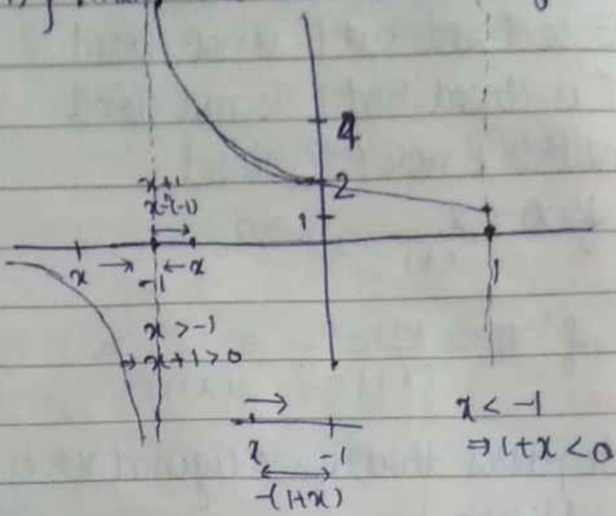
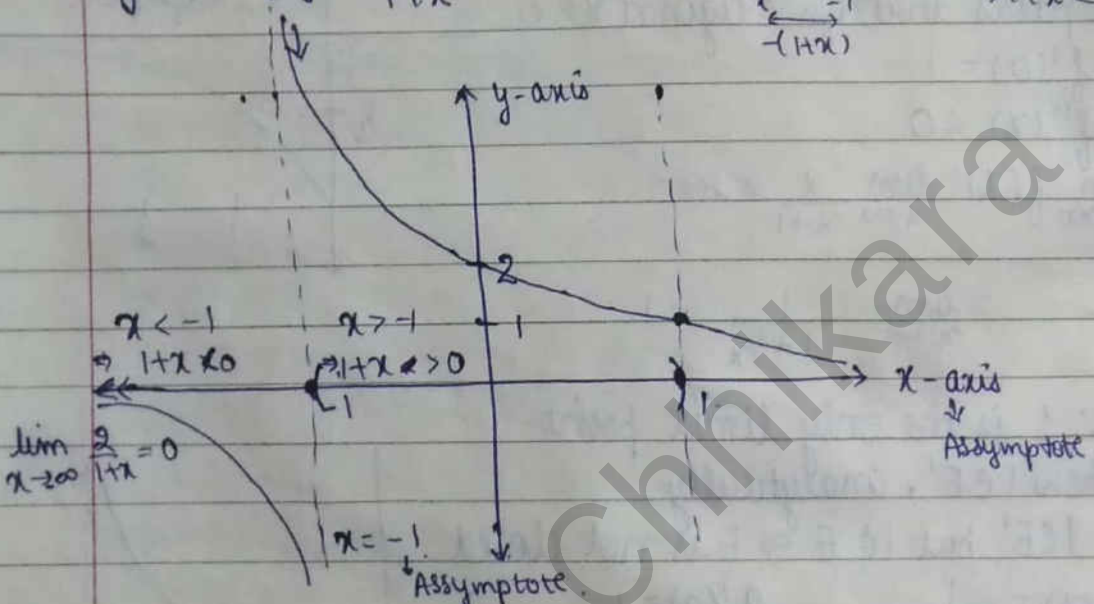
JAM Let $A = \left\{ \frac{2}{1+x} : x \in (-1,1) \right\}$. Then the derived set of A is ___?

Sol^n: $y = \frac{2}{1+x}$

$\frac{dy}{dx} = -\frac{2}{(1+x)^2} < 0$



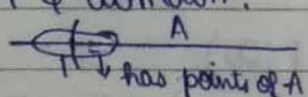
graph of $y = \frac{2}{1+x}$



$A = (1, \infty)$ as $x \in (-1, 1)$

If $x \in (-1, 1]$, then $A = [1, \infty)$ and if $x \in [-1, 1]$, then question is not valid as $-1 \notin$ domain.

$A' = [1, \infty)$



JAM Let $Y = \left\{ \frac{x}{1+|x|} : x \in \mathbb{R} \right\}$. Then $Y' =$ ___?

Sol^n: $y = \frac{x}{1+|x|} = \begin{cases} \frac{x}{1+x} & ; x \geq 0 \\ \frac{x}{1-x} & ; x < 0 \end{cases}$

Odd func^n

$y = \frac{x}{x+1}$, $x \geq 0 \Rightarrow \frac{dy}{dx} = \frac{1}{(x+1)^2} > 0$ & $\left.\frac{dy}{dx}\right|_{x=0} = 1$

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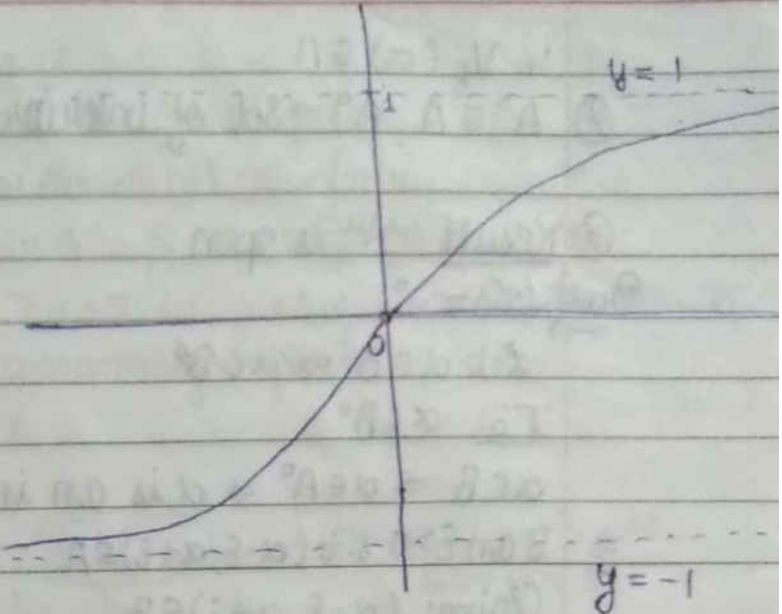
Contradiction in Real Analysis.



$\frac{d^2y}{dx^2} = \frac{-2}{(x+1)^3} < 0$

$\lim_{x\to\infty} \frac{x}{(x+1)} = \lim_{x\to\infty} \frac{1}{\frac{1}{x}+1}$

$= \frac{1}{0+1} = 1$

Now, $y = \frac{x}{1-x}, x<0$

$\frac{dy}{dx} = \frac{1}{(1-x)^2} > 0$

$\frac{d^2y}{dx^2} = \frac{-(-2)}{(1-x)^3} > 0$

S.S. of y: $-$, $+$, $-$; 0, 1

$\lim_{x\to-\infty} \frac{x}{1-x} = \lim_{x\to-\infty} \frac{1}{\frac{1}{x}-1} = -1$

$Y = (-1, 1) \Rightarrow Y' = [-1, 1]$

JAM Let A be an infinite subset of $\mathbb{R}$ such that $A \cap \mathbb{Q} = \phi$. Then

(a) A must have a limit point in $\mathbb{Q}$

(b) A must have a limit point in $\mathbb{R}\setminus\mathbb{Q}$

(c) A is not closed

✓(d) $\mathbb{R}\setminus A$ must have a limit point in $\mathbb{Q}$ (not only for $\mathbb{Q}$ but also for any n-e set)

Solⁿ: $A \subseteq \mathbb{Q}^c$ $(\because A \cap \mathbb{Q} = \phi) \Rightarrow \mathbb{R}\setminus A \supseteq \mathbb{R}\setminus\mathbb{Q}^c = \mathbb{Q}$

$A' = \phi \Rightarrow A$ is closed

Let $A = \{\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \dots\} = \{\sqrt{p} : p \text{ is prime}\}$ has no limit point.

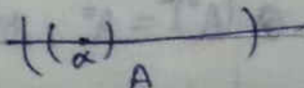
So, $\mathbb{Q} \subseteq \mathbb{R}\setminus A$

$\Rightarrow \mathbb{Q}' \subseteq (\mathbb{R}\setminus A)' \Rightarrow \mathbb{R} \subseteq (\mathbb{R}\setminus A)' \Rightarrow \mathbb{R} = (\mathbb{R}\setminus A)'$

So, every real number is its limit point.

★ $A \subseteq \mathbb{R}$, collect all the interior points of A

α is an interior point if $\exists$ some ϵ-nbhood $V_\epsilon(\alpha)$ of α such that $V_\epsilon(\alpha) \subseteq A$



Is it possible that an interior point α of A doesn't belong to A?

No

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$\because V_\varepsilon(\alpha) \subseteq A$

⊛ $A^\circ \subseteq A$, $A^\circ \rightarrow$ Set of interior points of α

⊛ **Result:** A° is open.

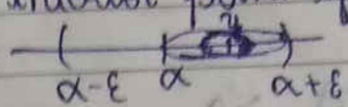
Proof: Let $B = A^\circ$.

Let $\alpha \in B$ ~~$\Rightarrow \alpha \in B^\circ$~~

T.S: $\alpha \in B^\circ$

$\alpha \in B \Rightarrow \alpha \in A^\circ \Rightarrow \alpha$ is an interior point of A.

$\Rightarrow \exists$ an $\varepsilon > 0$ s.t. $(\alpha-\varepsilon, \alpha+\varepsilon) \subseteq A$



Claim: $(\alpha-\varepsilon, \alpha+\varepsilon) \subseteq B$

Let $y \in (\alpha-\varepsilon, \alpha+\varepsilon)$

Take $\varepsilon' = \min\{y-(\alpha-\varepsilon), y-(\alpha+\varepsilon)\}$

Then $(y-\varepsilon', y+\varepsilon') \subseteq (\alpha-\varepsilon, \alpha+\varepsilon) \subseteq A$

$\Rightarrow y \in A^\circ$ i.e. $y \in B$

$\therefore (\alpha-\varepsilon, \alpha+\varepsilon) \subseteq \underset{\Downarrow \atop \alpha \in B^\circ}{B} \Rightarrow B$ is open i.e. A° is open

★ 

$A^\circ \cup \{\gamma\} \rightarrow$ Not open

⊛ **Result:** A° is the largest open set in A.

Proof: $A^\circ \subseteq A$ and A° is open $\rightarrow$ Done.

S: Open set in A

Claim: $S \subseteq A^\circ$

Let $\alpha \in S \Rightarrow \exists\, \varepsilon > 0$ s.t. $(\alpha-\varepsilon, \alpha+\varepsilon) \subseteq S$

Consequently, $(\alpha-\varepsilon, \alpha+\varepsilon) \subseteq A$ ($\because S \subseteq A$)

$\Rightarrow \alpha \in A^\circ$

⊛ A is open $\Leftrightarrow A^\circ = A$ ($\because A^\circ$ is the largest open set in A)

⊛ $(A^\circ)^\circ = A^\circ$; since A° is open

JAM 2015 Let $A \subset \mathbb{R}$, $A \neq \phi$ and $I(A)$ be the collection of all interior

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points of A. Then $I(A)$ can be

(a) empty (b) singleton

(c) finite set with more than one element (d) countably infinite

Sol^n (a) If A is finite, then $I(A) = A^\circ = \phi$

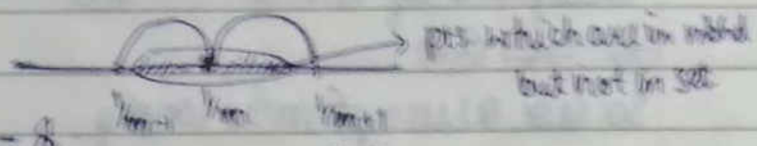
If $O \neq \phi$, O is open, then $\exists\ \alpha \in O$ s.t. for any $\epsilon > 0$ $(\alpha-\epsilon, \alpha+\epsilon) \subseteq O$

$|(\alpha-\epsilon, \alpha+\epsilon)| = |\mathbb{R}| = c \rightarrow$ uncountably infinite

So, neither (b), (c) nor (d).

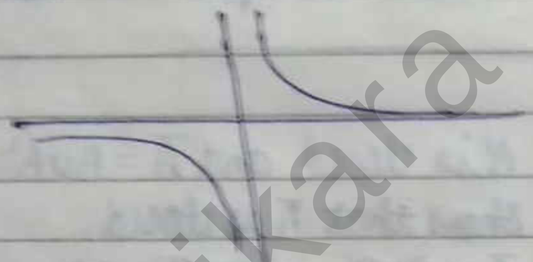
* ① $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$

$\Rightarrow$ A is not open, so, $A^\circ = \phi$



② $B = \left\{ \frac{1}{x} : x \in \mathbb{R}^* \right\}$

$(-\infty, 0) \cup (0, \infty)$ is open as union of open set is open



$\alpha \in (-\infty, 0)$

Set $\epsilon = |\alpha|$

$(\alpha - |\alpha|, \alpha + |\alpha|) \subseteq (-\infty, 0)$

So, B is open $\Rightarrow B^\circ = B$

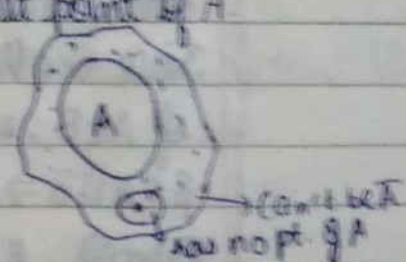
• Closure Point of a set $A \subseteq \mathbb{R}$

Closure of A $\leftarrow \bar{A} := A \cup A'$

Element of $\bar{A}$ closure points of A or adherent point

either an element of A or a limit point of A

* $A \subseteq \bar{A}$



Q- Show that A' is closed.

Sol^n: Let $B = A'$

Suppose $\alpha \in B'$

T.S: $\alpha \in B$

Let, if possible $\alpha \notin B$ i.e. $\alpha \notin A'$

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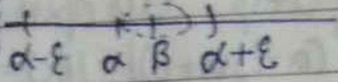
$X \subseteq Y$
$\alpha \in X \Rightarrow \alpha \in Y$ or $\alpha \notin Y \Rightarrow \alpha \notin X$

$\Rightarrow$ α is not a limit point of A.
There exists an $\varepsilon_0 > 0$ s.t. $(\alpha-\varepsilon_0, \alpha+\varepsilon_0)$ has no points of A, other than α. — ⊛
As $\alpha \in B'$, so $(\alpha-\varepsilon_0, \alpha+\varepsilon_0)$ has an element, say β of B, where $\alpha \neq \beta$
Let $\varepsilon' = \min\{|\beta-\alpha|, |\beta-(\alpha+\varepsilon)|, |\beta-(\alpha-\varepsilon)|\}$



$\beta \in B, \beta \in A'$
i.e. $\beta \in A' \Rightarrow (\beta-\varepsilon', \beta+\varepsilon')$ contains a point of A other than β
that point of $A \neq \alpha$ — ⊛⊛
⊛ $\Rightarrow\Leftarrow$ ⊛⊛
So, our assumption is wrong.

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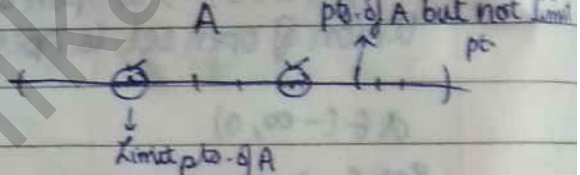
- A' is closed and $\bar{A} = A \cup A'$.

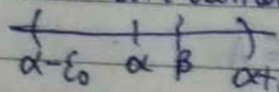
Show that $\bar{A}$ is closed.

Proof: $\bar{A} = A \cup B$, where $B = A'$



T.S: $(\bar{A})' \subseteq \bar{A}$

Claim: $(B)' = (A \cup B)'$
$B \subseteq A \cup B \Rightarrow B' \subseteq (A \cup B)'$ $(\because X \subseteq Y \Rightarrow X' \subseteq Y')$
We show that $(A \cup B)' \subseteq B'$
T.S: $\alpha \notin B' \Rightarrow \alpha \notin (A \cup B)'$
Suppose $\alpha \notin B'$ — ①
Let if possible, $\alpha \in (A \cup B)'$
① $\Rightarrow \exists$ an $\varepsilon_0 > 0 \ni (\alpha-\varepsilon_0, \alpha+\varepsilon_0)$ contains no element of B, other than α. — ⊛
Since $\alpha \in (A \cup B)'$, so $(\alpha-\varepsilon_0, \alpha+\varepsilon_0)$ contains an element, say β of $A \cup B$, where $\alpha \neq \beta$



$\beta \in A \cup B \Rightarrow \beta \in A$ or $\beta \in B$
From ⊛, $\beta \notin B$
As B is closed, $B' \subseteq B \Rightarrow \beta \notin B'$

Claim: $(A \cup B)' \subseteq B$, where $B = A'$
T.S: Let $\alpha \in (A \cup B)' \Rightarrow \alpha \in B$
Let $\alpha \notin B$. We show that $\alpha \notin (A \cup B)'$

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Notation of Open set : O or G Σ - Sum - Union
& closed set : F δ - Intersection

F_σ : union of closed sets & G_δ - Intersection of open sets

Let if possible, $\alpha \in (A \cup B)'$
$\alpha \notin B = A' \Rightarrow \exists$ an ε_0 s.t. $(\alpha-\varepsilon_0, \alpha+\varepsilon_0)$ contains no element of A other than α
Since $\alpha \in (A\cup B)'$, $(\alpha-\varepsilon_0,\alpha+\varepsilon_0)$ has an element β of $A\cup B$ other than α
We are sure $\beta \notin A$, so $\beta \in B = A'$
Let $\varepsilon' = \min\{|\alpha-\beta|, |\beta-(\alpha-\varepsilon_0)|, |\beta-(\alpha+\varepsilon_0)|\}$
So, $(\beta-\varepsilon', \beta+\varepsilon')$ has an element of A, different from β
Imp ⊛ $\bar{A}$ is closed. And $(\beta-\varepsilon', \beta+\varepsilon') \subseteq (\alpha-\varepsilon_0, \alpha+\varepsilon_0)$, so, it has an element of A, other than β and also other than α $\rightarrow\!\times$ (A)

$\alpha-\varepsilon_0$ α β $\alpha+\varepsilon_0$
It has no element of A other than α. and has an element of $A\cup B$ other than α

Q- Find $\bar{A}$, if
① $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ ② $A = \mathbb{Q}$
③ $A = \{\frac{1}{3^n} : n \in \mathbb{N}\}$

Solⁿ: ① $A' = \{0\}$
$\bar{A} = A \cup A' = \{\frac{1}{n} : n\in\mathbb{N}\} \cup \{0\}$

② $A' = \mathbb{R}$
$\bar{A} = A \cup A' = \mathbb{Q} \cup \mathbb{R} = \mathbb{R}$

③ $A' = \{0\}$
$\bar{A} = A \cup \{0\}$

⊛ Result: $\bar{A}$ is the smallest closed set containing A
Proof: B: closed contains A
(Pg-100) T.P: $\bar{A} \subseteq B$

- A set A is called an F_σ-set if A can be written as a countable union of closed sets

- A set A is called an G_δ-set if A can be written as a countable intersection of open sets

Q- Show $\mathbb{Q}$ is an F_σ-set
Solⁿ: $\mathbb{Q} = \bigcup_{x\in\mathbb{Q}} \{x\}$
$\{x\}$ is ~~union~~ closed set as every finite set is closed
and as $\mathbb{Q}$ is countable, $\therefore$ there are countable choices for $\{x\}$
$\therefore \bigcup_{x\in\mathbb{Q}}\{x\}$ is countable union of closed set

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$\mathbb{R}\setminus\{1\} \cap \mathbb{R}\setminus\{2\} \cap \mathbb{R}\setminus\{3\} = \mathbb{R}\setminus\{1,2,3\}$

$\therefore$ $\mathbb{Q}$ is an F_σ - set.

Q- Show that $\mathbb{R}\setminus\mathbb{Q}$ is a G_δ-set.

Solⁿ: $\mathbb{R}\setminus\mathbb{Q} = \underbrace{\bigcap_{x\in\mathbb{Q}}}_{\text{Countable intersection}} \mathbb{R}\setminus\{x\}$

$\mathbb{R}\setminus\{x\}$ is open as $\{x\}$ is closed ($\because$ complement of closed set is open)

Q- Show $[a,b]$ is a G_δ-set.

Solⁿ: $[a,b] = \underbrace{\bigcap_{n\in\mathbb{N}}}_{\text{Countable intersection}} \left(a-\frac{1}{n}, b+\frac{1}{n}\right)$

Q- Is ~~Show~~ $[a,b)$ a F_σ-set or G_δ-set?

Solⁿ: $[a,b) = \underbrace{\bigcap_{n\in\mathbb{N}}}_{\text{Countable intersection}} \left(a-\frac{1}{n}, b-\frac{1}{n}\right)$ $\Rightarrow [a,b)$ is G_δ - set

Imp.

- Compact sets (K): A set $K \subseteq \mathbb{R}$ is said to be compact if any sequence (a_n) in K has a convergent subsequence (a_{n_k}) which converges to a point in K.

  $(a_{n_k}) \rightarrow l \in K$.

Q- Show $[a,b]$ is compact.

Solⁿ: Let (a_n) be a sequence in $[a,b]$

By B.W. Thm, bounded sequence has a convergent subsequence

As (a_n) is bdd.

$\Rightarrow$ (a_n) has a convergent subsequence (a_{n_k}) (By B.W. Thm)

Let $(a_{n_k}) \rightarrow l$, $a \le a_{n_k} \le b$ $\Rightarrow a \le l \le b$

$\Rightarrow l \in \overline{[a,b]}$ ($\because a \in \bar{A} \Leftrightarrow \exists$ a seq. (a_n) in A s.t. $(a_n) \rightarrow a$, $a_n \neq a$)

$\Rightarrow l \in [a,b]$ [$\because [a,b]$ is closed]

So, $[a,b]$ is compact.

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Payere - Mathematics in U.S.
J. J. Sylvester

Q.E.D. → Quot erat demonstradum

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(*) T.S: $\bar{A}$ is closed i.e. $(\bar{A})' \subseteq \bar{A}$

Let $x \in (\bar{A})'$ i.e. $x \in (A \cup B)'$, where $B = A'$ — ①

We've $(A \cup B)' \subseteq B$ — ②

From ① & ②, $x \in B$ i.e. $x \in A'$ — (*)

$\bar{A} = A \cup A'$ — (**)

From (*) & (**), $x \in \bar{A}$ $\Rightarrow (\bar{A})' \subseteq \bar{A}$

(*) Result: $\alpha \in \bar{A} \Leftrightarrow$ There exists a sequence (x_n) in $A \ni (x_n) \to \alpha$

Proof: $(\Rightarrow)$ Let $\alpha \in \bar{A}$ i.e. $\alpha \in A \cup A'$

$\Rightarrow \underbrace{\alpha \in A}_{(\alpha,\alpha,\alpha,\ldots)\to\alpha}$ or $\underbrace{\alpha \in A'}_{\text{Our work is done}}$ [$\because \alpha \in A' \Leftrightarrow$ There exists a sequence (x_n) in $A \ni (x_n) \to \alpha$, where $x_n \neq \alpha \ \forall n$]

$(\Leftarrow)$ Suppose (x_n) is a sequence in A s.t. $(x_n) \to \alpha$

T.S: $\alpha \in \bar{A}$

T.S: Take any ε-nbhd, $V_\varepsilon(\alpha)$ of α, i.e., $(\alpha-\varepsilon, \alpha+\varepsilon) \cap A \neq \phi$, for each $\varepsilon > 0$

As $(x_n) \to \alpha$, $\exists$ an $N \in \mathbb{N}$ s.t. $|x_n - \alpha| < \varepsilon \ \forall n \geq N$

i.e., $x_n \in (\alpha-\varepsilon, \alpha+\varepsilon) \ \forall n \geq N$

$x_n \in A \ \forall n \in \mathbb{N}$

$\square$ (Halmos' box) Proof is over

★ If $\alpha \in A'$, then each ε-n'hood of α contains a point of A other than α.

Ex (Pg 101)

(*) Result: $\alpha \in \bar{A} \Leftrightarrow$ Each neighborhood of α contains a point of A

Q- Show ① $\overline{A \cup B} = \bar{A} \cup \bar{B}$ ② $(A \cup B)' = A' \cup B'$

③ $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$ ④ $(A \cap B)^\circ = A^\circ \cap B^\circ$ ⑤ $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$

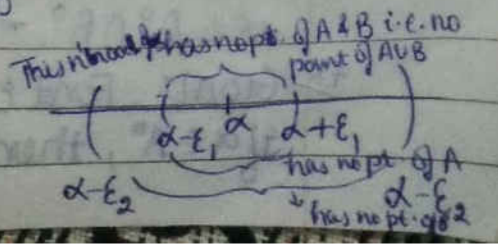
★ $X \subseteq Y \Rightarrow X' \subseteq Y'$] $\Rightarrow X \cup X' \subseteq Y \cup Y'$ i.e. $\bar{X} \subseteq \bar{Y}$

Sol^n: ① $A \subseteq A \cup B \Rightarrow \bar{A} \subseteq \overline{A \cup B}$

$B \subseteq A \cup B \Rightarrow \bar{B} \subseteq \overline{A \cup B}$] $\Rightarrow \bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$

We'll show $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$

Let $\alpha \notin \bar{A} \cup \bar{B} \Rightarrow \alpha \notin \bar{A}$ & $\alpha \notin \bar{B}$



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$A \subseteq \mathbb{R}$: open $\Leftrightarrow \mathbb{R}\setminus A$ is closed
$\mathbb{R}$: open $\Rightarrow \mathbb{R}\setminus\mathbb{R}$ is closed i.e. $\emptyset$ is closed
$\emptyset \to$ Empty set $\phi \to$ Fie

As $\alpha \notin \bar{A} \Rightarrow \exists$ an $\varepsilon_1 > 0 \ni (\alpha-\varepsilon_1, \alpha+\varepsilon_1)$ has no pt. of A
and $\alpha \notin \bar{B} \Rightarrow \exists$ an $\varepsilon_2 > 0 \ni (\alpha-\varepsilon_2, \alpha+\varepsilon_2)$ has no pt. of B.
$\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$
$(\alpha-\varepsilon, \alpha+\varepsilon)$ has no point of $A\cup B$
$\Rightarrow \alpha \notin \overline{A\cup B}$

(2) $A \subseteq A\cup B \Rightarrow A' \subseteq (A\cup B)'$ & $B \subseteq A\cup B \Rightarrow B' \subseteq (A\cup B)' \Rightarrow A'\cup B' \subseteq (A\cup B)'$
We'll show $(A\cup B)' \subseteq A'\cup B'$
Let $\alpha \notin A'\cup B' \Rightarrow \alpha \notin A'$ & $\alpha \notin B'$
As $\alpha \notin A' \Rightarrow \exists$ an $\varepsilon_1 > 0 \ni (\alpha-\varepsilon_1, \alpha+\varepsilon_1)$ has no pt. of A
& $\alpha \notin B' \Rightarrow \exists$ an $\varepsilon_2 > 0 \ni (\alpha-\varepsilon_2, \alpha+\varepsilon_2)$ has no pt. of B
$\varepsilon = \min\{\varepsilon_1, \varepsilon_2\} \Rightarrow (\alpha-\varepsilon, \alpha+\varepsilon)$ has no point of $A\cup B \Rightarrow \alpha \notin (A\cup B)'$

(3) Provide an example of A, B with $\overline{A\cap B} \neq \bar{A}\cap\bar{B}$ i.e. $\overline{A\cap B} \subsetneq \bar{A}\cap\bar{B}$
[(*) A is closed $\Leftrightarrow \bar{A} = A$ ($\because \bar{A} = A\cup A'$ & $A' \subseteq A$)]
Let $A = (1,2) \Rightarrow \bar{A} = [1,2]$
$B = (2,3) \Rightarrow \bar{B} = [2,3]$
$\bar{A}\cap\bar{B} = \{2\}$
$A\cap B = \cancel{\phi}\ \phi \Rightarrow \overline{A\cap B} = \bar{\phi} = \phi$ ($\because \phi$ is closed)
$\therefore \overline{A\cap B} \neq \bar{A}\cap\bar{B}$
T.S. $\overline{A\cap B} \subseteq \bar{A}\cap\bar{B}$

$$\left.\begin{array}{l} A\cap B \subseteq A \Rightarrow \overline{A\cap B} \subseteq \bar{A} \\ \& \ A\cap B \subseteq B \Rightarrow \overline{A\cap B} \subseteq \bar{B} \end{array}\right\} \Rightarrow \overline{A\cap B} \subseteq \bar{A}\cap\bar{B}$$

(4) T.S. $(A\cap B)^\circ \subseteq A^\circ \cap B^\circ$
As $X \subseteq Y \Rightarrow X^\circ \subseteq Y^\circ$, so,

$$\left.\begin{array}{l} A\cap B \subseteq A \Rightarrow (A\cap B)^\circ \subseteq A^\circ \\ A\cap B \subseteq B \Rightarrow (A\cap B)^\circ \subseteq B^\circ \end{array}\right\} \Rightarrow (A\cap B)^\circ \subseteq A^\circ \cap B^\circ$$

We show, $A^\circ \cap B^\circ \subseteq (A\cap B)^\circ$
$\alpha \in A^\circ \cap B^\circ \Rightarrow \alpha \in A^\circ$ & $\alpha \in B^\circ$
GOAL: Find $\varepsilon > 0$ s.t. $(\alpha-\varepsilon, \alpha+\varepsilon) \subseteq A\cap B$
If $\alpha \in A^\circ$, then $\exists\, \varepsilon_1 > 0$ s.t. $(\alpha-\varepsilon_1, \alpha+\varepsilon_1) \subseteq A$

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If $\alpha \in B^\circ$, then $\exists\, \varepsilon_2 > 0$ s.t. $(\alpha - \varepsilon_2, \alpha + \varepsilon_2) \subseteq B$

$\varepsilon = \min(\varepsilon_1, \varepsilon_2)$

$(\alpha - \varepsilon, \alpha + \varepsilon) \subseteq A \cap B$

$\Rightarrow \alpha \in (A \cap B)^\circ \Rightarrow A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$

(5) Provide an example of A & B s.t. $(A^\circ \cup B^\circ) \subsetneq (A \cup B)^\circ$

Let $A = [1,2] \Rightarrow A^\circ = (1,2)$ & $B = [2,3] \Rightarrow B^\circ = (2,3)$

$A^\circ \cup B^\circ = (1,2) \cup (2,3)$

$(A \cup B) = [1,3] \Rightarrow (A \cup B)^\circ = (1,3)$

Clearly, $A^\circ \cup B^\circ \subsetneq (A \cup B)^\circ$, hence, $A^\circ \cup B^\circ \neq (A \cup B)^\circ$

T.S: $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$

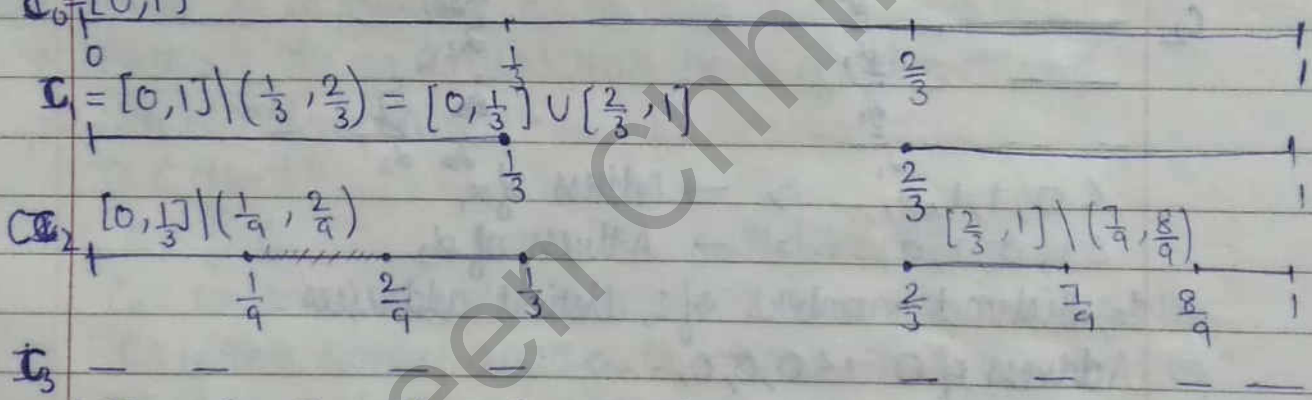
$$\left.\begin{array}{l} A \subseteq A \cup B \Rightarrow A^\circ \subseteq (A \cup B)^\circ \\ B \subseteq A \cup B \Rightarrow B^\circ \subseteq (A \cup B)^\circ \end{array}\right] \Rightarrow (A^\circ \cup B^\circ) \subseteq (A \cup B)^\circ$$

- $C_0 = [0,1]$

$C_1 = [0,1] \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$

C_2: $\left[0, \frac{1}{3}\right] \setminus \left(\frac{1}{9}, \frac{2}{9}\right)$ $\quad \left[\frac{2}{3}, 1\right] \setminus \left(\frac{7}{9}, \frac{8}{9}\right)$

C_3



$C_0 = [0,1]$, $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$

$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$

$\vdots$

C_n: Union of 2^n closed intervals
each interval with length $\frac{1}{3^n}$

$C = \bigcap_{n=0}^{\infty} C_n \to$ Cantor set

$C \neq \emptyset$ as C contains at least the end points, which are of the form $\frac{m}{3^n}$

Is C countable? No.

Does C contain open intervals? No

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Does C contain any irrational number? Yes

If C contains only end point, then $C \subseteq \mathbb{Q}$ ($\because$ end points are rational)

$\Rightarrow$ C is countable, but this is wrong as C doesn't contain only end pts

Total length removed $= \frac{1}{3} + 2\cdot\frac{1}{9} + 4\cdot\frac{1}{27} + \cdots$

$= \frac{1}{3} + 2\cdot\frac{1}{3^2} + 2^2\cdot\frac{1}{3^3} + 2^3\cdot\frac{1}{3^4} + \cdots$

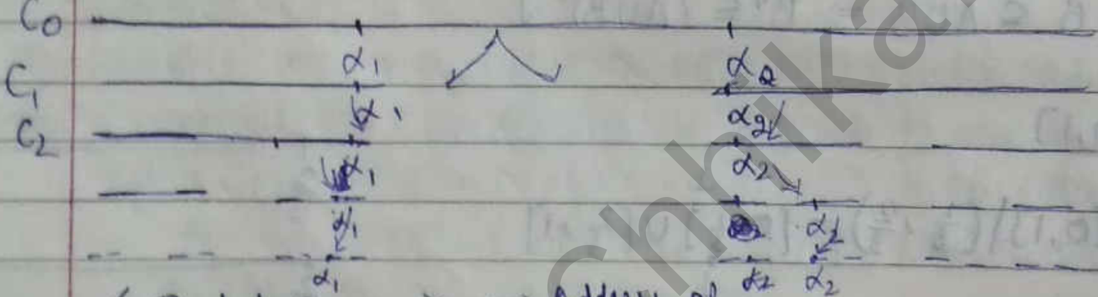
$= \frac{1}{3}\left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots\right]$

$r = \frac{2}{3} < 1$ $\left[a + ar + ar^2 + \cdots = \frac{a}{1-r},\ r<1\right]$

$= \frac{1}{3}\left[\frac{1}{1-\frac{2}{3}}\right] = \frac{1}{3}\left[\frac{3}{1}\right] = 1$

Total length = 1

(*) Cantor set has length zero → "measure"



$\langle 0,1,1,0,\ldots\rangle \rightarrow$ Address of α_1

$\langle 1,0,1,0,\ldots\rangle \rightarrow$ Address of α_2

So, Distinct members of C, distinct addresses.

Address of 0 $\rightarrow \langle 0,0,0,\ldots\rangle$

Address of 1 $\rightarrow \langle 1,1,1,\ldots\rangle$

Observe that addresses of end points are eventually constant sequences.

(*) # of sequences with terms 0 or 1 = # of elements of C

such sequences are uncountably many infact $\mathfrak{c}$ in "number"

$\therefore |C| = \mathfrak{c}$ $\qquad 2^{\aleph_0} = |\mathbb{R}|$

Length of the Cantor set is zero

Cardinality of the Cantor set is that of $\mathbb{R}$

As C is not countable, so, it has irrational numbers

$C \nsubseteq \mathbb{Q}$

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$(a,b) \subseteq C$?

Let $\varepsilon > 0$ s.t. $\varepsilon = |b-a|$

$\frac{1}{3^{m_0}} < \frac{1}{m_0} < \varepsilon$ (By Arch. Prop. $\exists\, m_0 \in \mathbb{N}$ s.t. $\frac{1}{m_0} < \varepsilon$)

↳ ~~cardi~~ Length of intervals.

$(a,b) \subseteq C_{m_0}$? No.

↳ has 2^{m_0} intervals of length $1/3^{m_0}$

And (a,b) has length greater than $\frac{1}{3^{m_0}}$ as $\varepsilon = |b-a|$.

$\Rightarrow (a,b) \notin \bigcap_{n=0}^{\infty} C_n = C$

$\therefore$ C has no open interval.

⊗ No open interval is contained in the cantor set.

$\alpha \in C$

Is α an interior point of C? No (Why?)

As if α is an interior point of C then ~~st~~ every ε-n'hood is contained in C but no ε-n'hood of α is contained in C which is impossible.

$\Rightarrow$ No point of C is an interior point of C, i.e., $C^\circ = \phi$

$\therefore$ C is not open.

Is C closed?

$C = \bigcap_{n=1}^{\infty} C_n$

C_n: union of 2^n $\underbrace{\text{closed intervals}}$ (FINITE; are closed sets) $\Rightarrow C_n$ is closed $\forall\, n \in \mathbb{N}$

C: intersection of closed sets $\Rightarrow$ C is closed

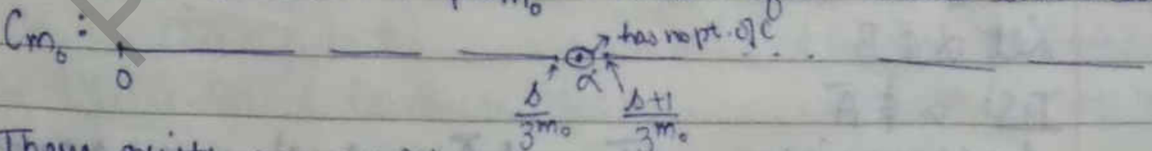
$C \subseteq [0,1] \Rightarrow C' \subseteq [0,1]$ ($\because [0,1]$ is closed set)

Let α ~~be~~ $\in [0,1]$, $\alpha \notin C$

i.e. $\alpha \notin \bigcap_{n \in \mathbb{N}} C_n$

$\Rightarrow \exists$ some $m_0 \in \mathbb{N}$ s.t. $\alpha \notin C_{m_0}$ → union of 2^{m_0} closed intervals

C_{m_0}:



There exists an $s \in \mathbb{N}$ s.t. $\frac{s}{3^{m_0}} < \alpha < \frac{s+1}{3^{m_0}}$

Is $\alpha \in C'$? No, as we get a n'hood of ε s.t. it has no pt of C other than α.

$\varepsilon := \min\left\{\alpha - \frac{s}{3^{m_0}}, \frac{s+1}{3^{m_0}} - \alpha\right\}$

$(\alpha-\varepsilon, \alpha+\varepsilon) \cap C = \phi$

$\Rightarrow \alpha \notin \bar{C} = C \cup C' \Rightarrow \alpha \notin C' \Rightarrow$ If any pt. doesn't belong to C then it is not its limit point. So, C is closed.

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Let $\alpha \in C$

Is $\alpha \in C'$?

$\alpha \in C \to \alpha \in C_n \;\forall n \in \mathbb{N}$

In particular, $\alpha \in C_{m_0} \Rightarrow \exists$ some $s \in \mathbb{N}$ s.t. $\frac{s}{3^{m_0}} \le \alpha < \frac{s+1}{3^{m_0}}$

C_{m_0} : —— —— $\alpha-\varepsilon$ $\alpha+\varepsilon$ α $s/3^{m_0}$ $s+1/3^{m_0}$ ...... ——

$C \subseteq [0$ where $\left[\frac{s}{3^{m_0}}, \frac{s+1}{3^{m_0}}\right] \subseteq C_{m_0}$

Given $\varepsilon > 0$

$\frac{1}{3^{m_0}} \leftarrow C_{m_0}$: $\alpha-\varepsilon$ $\alpha+\varepsilon$ α $\quad$ C_{m_0} : α

$\frac{1}{3^{m_0+1}} \leftarrow C_{m_0+1}$: α $\quad$ C_{m_0+1} : α

$\frac{1}{3^{m_0+2}} \leftarrow C_{m_0+2}$: α $\quad$ C_{m_0+2} :

$\therefore$ We can find end points (pts of C) arbitrary close to α.

$s \ni \frac{1}{3^{m_0+s}} < \varepsilon$

OR we can find $s_0 \in \mathbb{N}$ s.t. $\frac{1}{3^{s_0}} < \varepsilon$

C_{s_0} : length of intervals $\frac{1}{3^{s_0}}$

$\alpha \in C \Rightarrow \alpha \in C_{s_0} \Rightarrow \alpha$ is in one of 2^{s_0} closed intervals

Pick end pt β of that closed interval, $\alpha \neq \beta$

$|\alpha - \beta| < \frac{1}{3^{s_0}} < \varepsilon \Rightarrow \beta \in (\alpha-\varepsilon, \alpha+\varepsilon)$

β: member of C

(Pg-93)

Result: $\bar{A}$ is the smallest closed set containing A

Proof. $\bar{A}$ is closed and $A \subseteq \bar{A}$ (Done)

B: closed contains $A \Rightarrow A \subseteq B$

T.P. $\bar{A} \subseteq B$ i.e. $\alpha \notin B \Rightarrow \alpha \notin \bar{A}$

As B is closed $\Rightarrow B' \subseteq B$ —— ①

Let $\alpha \notin B$

T.S: $\alpha \notin \bar{A}$

Let if possible, $\alpha \in \bar{A}$ and $\bar{A} = A \cup A' \Rightarrow \alpha \in A$ or $\alpha \in A'$

Case I: If $\alpha \in A$, then $\alpha \in B$ ($\because A \subseteq B$) $\times$ ($\because \alpha \notin B$)

So, $\alpha \notin A$

Case II: If $\alpha \in A'$.

$A \subseteq B \Rightarrow A' \subseteq B'$

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So, $\alpha \in B' \subseteq B$ (By ①) $\Rightarrow \alpha \in B$ ※ ($\because \alpha \notin B$)

$\therefore$ Our assumption is wrong

$\alpha \notin \bar{A}$

$\therefore \bar{A} \subseteq B$.

(19-25)

※ **Result:** $\alpha \in \bar{A} \Leftrightarrow$ Each n'hood of α contains a point of A.

Proof: ($\Rightarrow$) $\alpha \in \bar{A}$

$\Rightarrow \exists$ a sequence (x_n) in A s.t. $(x_n) \longrightarrow \alpha$

Let $\varepsilon > 0$ be given.

$|x_n - \alpha| < \varepsilon \ \forall n \geq N \Rightarrow x_n \in (\alpha-\varepsilon, \alpha+\varepsilon) \ \forall n \geq N$

As $(x_n) \in A$ and $\varepsilon > 0$ is arbitrary, $\therefore$ each nbhood of α contains a point of A.

($\Leftarrow$) Let $\varepsilon > 0$ be given and each n'hood of α contains a point of A

$\Rightarrow (\alpha-\varepsilon, \alpha+\varepsilon)$ contains a point of A, say, β

$\left.\begin{array}{l}\text{If } \beta = \alpha, \text{ then } \alpha \in A \\ \& \text{ If } \beta \neq \alpha, \text{ then } \alpha \in A'\end{array}\right] \Rightarrow \alpha \in A \cup A' = \bar{A}$

$\therefore \alpha \in \bar{A}$

Real Analysis Test

Date: Sept 10, 2016
Topics: Countability of sets, Bounded sets, Sequences

"It does not matter how much knowledge we have, but it matters whether how much eager are we to gain that."-**Parveen Chhikara**

1. a, 2. b, 3. d, 4. a, 5. b, 6. c, 7. c, 8. c, 9. b, 10. b,
11.a,c, 12. a, b, d, 13. a, c, 14. b, d, 15. a, b, c,
16. b, c, 17. -03849, 18. 0.5, 19. 2.71, 20. 0

0. If in this test, you fail to do a lot of questions, then
(a) you should think that the test is tough, and you can not do anything.
(b) you lose your confidence.
(c) you think that you are very weak in studies.
(d) you do not lose your confidence, and try to give your best in the test.

Single-Correct Questions

1. Suppose that (x_n) is a convergent sequence and (y_n) is such that for any $\varepsilon > 0$, there exists $M \in \mathbf{N}$ such that $| x_n - y_n |< \varepsilon$ for all $n \geq M$. Then (y_n) is
(a) convergent.
(b) bounded but not convergent.
(c) bounded above but unbounded below.
(d) bounded below but unbounded above.

2.The set of the roots of all polynomial functions of degree 3, and with rational coefficients is
(a) uncountable.
(b) countable infinite.
(c) finite set with cardinality greater than 3.
(d) of cardinality 3.

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3. If $f : A \to B$ and the range of f is uncountable, then the domain of the function f
(a) may be countable.
(b) is countable.
(c) may be finite.
(d) is uncountable.

4. Suppose that f is continuous and that the sequence

$$x, f(x), f(f(x)), f(f(f(x))), \ldots$$

converges to l. Then
(a) $f(l) = l$.
(b) $f(l) = l^2$.
(c) $f(l) = \frac{1}{l}$.
(d) $f(l)$ does NOT exist.

5. The sequence $\{\frac{2n+1}{2n} : n \in \mathbf{N}\}$ is
(a) unbounded above.
(b) bounded.
(c) divergent
(d) unbounded below.

6. If u is an upper bound of a set A of real numbers and $u \in A$, then u is
(a) an infimum of A.
(b) both infimum and supremum of A.
(c) a supremum of A.
(d) neither infimum nor supremum of A.

7. Point out the WRONG statement out of the following.
(a) The countable union of countable sets is countable.
(b) If A and B are countable, then $A \times B$ is countable.
(c) The uncountable union of finite sets is countable.
(d) Every infinite set is equivalent to one of its proper subsets.

8. Given the sequence $\langle \frac{n}{n+1} \rangle$ and an arbitrary small positive number ε. Then the value of a positive integer m such that $\mid \frac{n}{n+1} - 1 \mid < \varepsilon$ whenever $n \geq m$ must satisfy

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(a) $m \leq \frac{1}{\varepsilon} - 1$.
(b) $m < \frac{1}{\varepsilon} - 1$.
(c) $m > \frac{1}{\varepsilon} - 1$.
(d) $m \geq \frac{1}{\varepsilon} - 1$.

9. Let $\lim \frac{s_n - 1}{s_n + 1} = 0$, then $\lim s_n$ equals
(a) 0.
(b) 1.
(c) -1.
(d) 2.

10. Which among the following is CORRECT?
(a) If a sequence of positive real numbers is not bounded, then the sequence diverges to infinity.
(b) If a sequence converges, then it is bounded.
(c) If a sequence is monontonically increasing, and bounded above, then it may fail to be convergent.
(d) Every bounded sequence is convergent.

Multi-Correct Questions

11. Which of the following statements is(are) TRUE?
(a) An infinite set contains a countable subset.
(b) If A is an infinite set and $x \in A$, then A and $A \setminus \{x\}$ are not equivalent.
(c) The intervals $(0, 1)$ and $[0, 1]$ are equivalent.
(d) The set of all ordered pairs of integers is not countable.

12. If $L \in \mathbf{R}$, $M \in \mathbf{R}$ and $L \leq M + \varepsilon$ for every $\varepsilon > 0$, then which of the following MAY be true?
(a) $L < M$.
(b) $L = M$.
(c) $L > M$.
(d) $L \leq M$.

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13. If (s_n) is a sequence of real numbers and if, for every $\varepsilon > 0$,

$$| s_n - L |< \varepsilon \text{ for every } n \geq N,$$

where N does not depend on ε, then
(a) finitely many terms are not equal to L.
(b) all but finitely many terms are equal to L.
(c) the terms which are equal to L are infinitely many.
(d) the terms which are equal to L are finitely many.

14. Which of the following statements is (are) TRUE for a sequence (s_n)?
(a) $(|s_n|)$ converges to $a \Leftrightarrow (s_n)$ converges to a.
(b) (s_n) converges to $a \Leftrightarrow (|s_n|)$ converges to $|a|$.
(c) $(|s_n|)$ converges to $a \Rightarrow (s_n)$ converges to a.
(d) $(|s_n|)$ converges to $0 \Leftrightarrow (s_n)$ converges to 0.

15. Let $s_1 > s_2$, and let $s_{n+1} = \frac{1}{2}(s_n + s_{n-1}), (n \geq 2)$. Then
(a) $s_1, s_3, s_5, \ldots$ is nonincreasing.
(b) $s_2, s_4, s_6, \ldots$ is nondecreasing.
(c) $(s_n)_{n=1}^{\infty}$ is a convergent sequence.
(d) $(s_n)_{n=1}^{\infty}$ is a divergent sequence.

16. If $\{s_n\}$ is a Cauchy sequence of real numbers which has a subsequence converging to L, then
(a) $\{s_n\}$ may not be convergent.
(b) $\{s_n\} \to L$.
(c) $\{s_n\}$ is bounded.
(d) $\{s_n\}$ is unbounded.

Numerical-Answer Type Questions

17. The infimum of the set $\{x^3 - 6x^2 + 11x - 6 : x \geq 1\}$ upto three decimal points is ....

18. $\lim_{n\to\infty} \frac{2n^3+5n}{4n^3+n^2} =$..............

19. The limit superior of the sequence $\{(1 + \frac{1}{n})^n\}$ is ....

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20. If $s_n = \frac{5^n}{n!}$, then $\lim_{n\to\infty} s_n =$ ....

`Best Wishes from Parveen Chhikara...`

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17/9/16

Test Discussion (10/9/16)

1. (x_n): convergent seq. $\Rightarrow (x_n) \longrightarrow l$

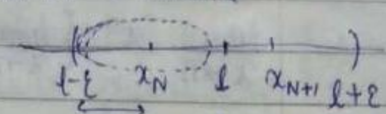
given $\varepsilon > 0$, $\exists\, M \in \mathbb{N}$ s.t. $|x_n - y_n| < \varepsilon \;\forall\; n \geq M$

Claim: $(y_n) \rightarrow l$

As $(x_n) \rightarrow l$, $\varepsilon' > 0$ given $\exists\, N \in \mathbb{N} \ni |x_n - l| < \varepsilon \;\forall n \geq N$

$\Rightarrow x_n \in (l-\varepsilon, l+\varepsilon)$

$|x_N - y_N| < \varepsilon''$



$\varepsilon'' = \min\{x_N - (l-\varepsilon), x_N - (l+\varepsilon)\}$

So, (y_n) is cgt. & hence bounded

2. $a_0 + a_1 x + a_2 x^2 + a_3 x^3$, $a_0, a_1, a_2, a_3 \in \mathbb{Q}$

$\aleph_0 \times \aleph_0 \times \aleph_0 \times \aleph_0 = \aleph_0^4 = \aleph_0 \Rightarrow$ Countably many poly^{ns}.

$\Rightarrow \aleph_0$ polyn & each has at most 3 roots

Roots $= \underbrace{3+3+\cdots}_{3 \cdot \aleph_0 = \aleph_0}$

⊛ $\aleph_0^k = \aleph_0$, $k < \infty$

$k\aleph_0 = \aleph_0$, $k < \infty$

3. $f: A \to B$, $R(f)$: Uncountable

Let, if possible, $A = \{a_1, a_2, \ldots\}$ be countable.

$R(f) = \{f(a_k) : k \in \mathbb{N}\} \Rightarrow |R(f)| \le \aleph_0$ R(f) is singleton set

$\to |R(f)|$ is smallest when f is constant func^n & max. when f is one-to-one, then $|R(f)| = |A| = \aleph_0 \to$ max. size of $R(f)$

$\Rightarrow R(f)$ is countable *

So, A is uncountable.

V. Imp * If f is a function, which is continuous at $x = a$ and $(x_n) \to a$, then $f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n)$.

"Continuous functions commute with limits"

4. $x, f(x), f(f(x)), f(f(f(x))), \ldots \to l$

$x, f^1(x), f^2(x), f^3(x), \ldots \to l$

$\Rightarrow (f^n(x)) \to l$ (Ignoring 1^{st} term i.e. x)

$\Rightarrow f(f^{n-1}(x)) \to l$

i.e. $\lim f^n(x) = l$ & $\lim f(f^{n-1}(x)) = l$

$\Rightarrow f(\lim f^{n-1}(x)) = l$ ($\because f$ is continuous)

* If $\lim x_n = l$, then $\lim x_{n-1} = l$

$\Rightarrow f(l) = l$

* $\lim_{x\to 2} [x] \ne [\lim_{x\to 2} x]$ b/c [ ] is not continuous at $x = 2$.

$\lim_{x\to 2}[x]$ doesn't exist; $[\lim_{x\to 2} x] = 2$

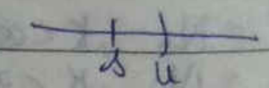
[ ] not cont. at integer pts and cont. at non integer pts.

$\lim_{x\to\pi} [x] = [\lim_{x\to\pi} x]$? Yes, [ ] is continuous @ π

5. $\left\{\frac{2n+1}{2n}\right\}$ is cgt., so, it is bounded

6. u: upper bound of A & $u \in A$.

If $s < u$, then can s be an U.B. of A?



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No, so: u is Sup. (A)
If A is singleton, then (b) is also true.

7.(b) A, B: countable, so, $A=\{a_1, a_2, \dots\}$ & $B=\{b_1, b_2, \dots\}$

$$A\times B = \begin{cases} (a_1,b_1), (a_1,b_2), (a_1,b_3), \dots \\ (a_2,b_1), (a_2,b_2), (a_2,b_3), \dots \\ (a_3,b_1), (a_3,b_2), (a_3,b_3), \dots \\ \vdots \end{cases}$$

$\mathbb{N} \to A\times B$ is one-to-one correspondence, so, $A\times B$ is countable
If, A, B are countable, then $A\times B$ is also countable.

(*) If $A_1, A_2, \dots, A_n$ are finitely many countable sets, then there cartesian product $\underbrace{A_1\times A_2 \times \dots \times A_n}_{\prod_{i=1}^{n} A_i}$ is also countable

* $A_1, A_2, A_3, \dots, A_n, \dots$: Infinitely many sets (Countably)
&, where $|A_i|=2$
$A_1\times A_2 \times A_3 \times \dots \times A_n \times \dots = \{(a_1, a_2, \dots, a_n, \dots) : a_i \in A_i\}$
2 choices

$$|\prod_{i\in\mathbb{N}} A_i| = 2^{\aleph_0} = c$$

→ If $A_1, A_2, \dots, A_n, \dots$ are countable sets, then $\prod_{i\in\mathbb{N}} A_i$ is uncountable, so,

(*) The cartesian product of countable number of countable sets is uncountable.

(a) countable union means union of countable number of sets is countable ↑
Countable cartesian product → cartesian product of countable no. of sets is not countable

(c) Uncountable union of finite sets is not countable.
$\underbrace{\bigcup_{x\in\mathbb{R}} \{x\}}_{\text{uncountable union of finite sets}} = \mathbb{R} \to$ not countable

(d) A : infinite set
Pick an element, say a_1, from A,

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$A\setminus\{a_1\} \to$ is it finite? No

Pick an element, say a_2 from $A\setminus\{a_1\}$, Observe $a_2 \neq a_1$

$A\setminus\{a_1, a_2\} \to$ is it finite? No

Pick an element, say a_3 from $A\{a_1, a_2\}$, Observe $a_3 \neq a_2 \neq a_1$

$A\setminus\{a_1, a_2, a_3\}$ is also not finite

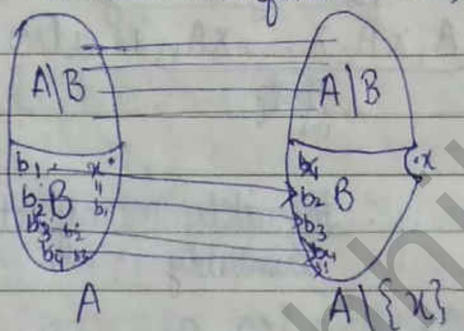
$\vdots$

$\{a_1, a_2, a_3, \ldots\} \subseteq A$ and is countable.

⊛ Result: An infinite set has a countable infinite subset.

★ A: infinite set (countable or uncountable)

A has a countable infinite subset, say B, $B = \{b_1, b_2, b_3, \ldots\}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \overset{\shortparallel}{x}$



$b_n \mapsto b_{n+1}$

$f: A \longrightarrow A\setminus\{x\}$ defined by $f(a) = \begin{cases} b_{n+1} & ; a = b_n \\ a & ; a \neq b_n \text{ i.e. } a \notin B \end{cases}$

is a bijection

So, $|A| = |A\setminus\{x\}|$

If B is finite, let $|B| = 100$, then $|B\setminus\{x\}| = 99$ then at least 2 elements have same image, then f is not a bijection.

$A \sim A\setminus\{x\}$

"Removing finitely many elements from an infinite set doesn't alter its cardinality"

11.(c) $(0,1) = [0,1]\setminus\{0,1\} \Rightarrow (0,1) \sim [0,1]$

(d) $\mathbb{Z} \times \mathbb{Z} \to$ Cartesian product of 2 countable sets is countable

(b) Proved above.

8. $\left(\frac{n}{n+1}\right) \longrightarrow 1$

given: $\varepsilon > 0$, $\exists$ an $N \in \mathbb{N} \ni$

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$\left|\frac{n}{n+1} - 1\right| < \varepsilon \; \forall \, n \geq N$ i.e. $\frac{1}{n+1} < \varepsilon \; \forall \, n \geq N$

i.e. $n > \frac{1}{\varepsilon} - 1$

So, choose $N > \frac{1}{\varepsilon} - 1$

If N: any integer $> \frac{1}{\varepsilon} - 1$ then $n \geq N \Rightarrow n > \frac{1}{\varepsilon} - 1$

9. Use Algebraic Limit Theorem.

10 (a) $(1, 2, 1, 3, 1, 4, \ldots) \rightarrow$ oscillating finitely not diverges to ∞

(d) $(\cancel{0}, 1, -1, +1, -1, \ldots) \rightarrow$ Bounded but not convergent.

12. Given $\varepsilon > 0$

$L \leq M + \varepsilon \Rightarrow L - M \leq \varepsilon$ — ⊛

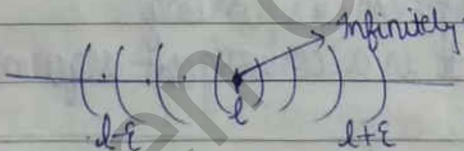
Can $L - M > 0$? No

Let it be, choose $\varepsilon = \frac{L-M}{2}$

Then from ⊛ $L - M \leq \frac{L-M}{2}$ ⇒⇐

So, $L \leq M$

13. $(s_n) \not\rightarrow l$
$\left(l - \frac{1}{n}\right)$



$(a_1, a_2, \ldots, a_{N-1}) \rightarrow$ may be outside or inside $(l-\varepsilon, l+\varepsilon)$

If N doesn't depend on ε, then the sequence is eventually constant sequence.

⊛ ~~If~~ for a convergent sequence, if N does NOT depend on $\varepsilon > 0$, then the sequence must be eventually constant.

# of terms that are not equal to $l \leq N-1$

N-1 may be zero, then all terms are equal to l then the seq is constant sequence.

14 (a) $(|s_n|) \rightarrow a \not\Rightarrow (s_n) \rightarrow a$, e.g. $s_n = (1, -1, 1, -1, \ldots)$

(b) same as above.

(d) $(|s_n|) \rightarrow 0 \Leftrightarrow (s_n) \rightarrow 0$

$(s_n) \rightarrow a \Rightarrow (|s_n|) \rightarrow |a|$, but converse is not true.

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If we write only $|a_2| > |a_1|$, then seq. becomes bounded

e.g. $1 \quad 2-\frac{1}{n} \quad 2$

We prove $(|s_n|) \to 0 \Rightarrow (s_n) \to 0$

given: $\varepsilon > 0$

GOAL: $|s_n - 0| < \varepsilon$ i.e. $|s_n| < \varepsilon$ i.e. $||s_n|| < \varepsilon$ i.e. $||s_n| - 0| < \varepsilon$

which is true.

15. $s_2 \quad s_4 \quad s_6 \quad s_5 \quad s_3 \quad s_1$

$\ldots < s_5 < s_3 < s_1$ & $s_2 < s_4 < s_6 < \ldots$ } Monotone & Bounded $\Downarrow$ Convergent seq.

Monotone dec. Monotone inc.

$\therefore (s_{2n})$ converges & (s_{2n+1}) converges.

Suppose $(s_{2n}) \to l_1$ & $(s_{2n+1}) \to l_2$

Let, if possible, $l_1 \neq l_2$

$$s_{n+1} = \frac{1}{2}(s_n + s_{n-1})$$

$$\Rightarrow s_{2m+1} = \frac{1}{2}(s_{2m} + s_{2m-1})$$

$$\Rightarrow \lim s_{2m+1} = \frac{1}{2}[\lim s_{2m} + \lim s_{2m-1}]$$

$$\Rightarrow l_2 = \frac{1}{2}(l_1 + l_2) \quad \Rightarrow\Leftarrow$$

So, our assumption is wrong.

Hence, $l_1 = l_2 \Rightarrow$ It is a convergent sequence

18/9/16 ⊛ Result: A set K is compact. $\Leftrightarrow$ K is closed & bounded

Characterization of compact set

Proof. ($\Rightarrow$) Let K be compact.

[Heine-Borel Theorm]

T.S: K is closed & bounded

Let if possible, K be not bounded.

[How to show that a set K is NOT compact?
We search a sequence in K whose no subseq. converges in K.]

Pick an element, say, $a_1 \in K$ s.t. $|a_1| > 1$ — makes unbounded

Pick an element, say, $a_2 \in K$ s.t. $|a_2| > 2$, $|a_2| > |a_1|$ → helps in making increasing seq

Pick an element, say, $a_3 \in K$ s.t. $|a_3| > 3$, $|a_3| > |a_2|$

$\vdots$

$(|a_n|)$: strictly increasing & unbounded.

Any subsequence of (a_n), that is unbounded

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$\Rightarrow$ No subseq is convergent

$\Rightarrow$ K is not compact. $\times$

So, K is bounded.

Suppose $\alpha \in K' \Rightarrow \exists$ a sequence (x_n) in K s.t. $(x_n) \to \alpha$, $x_n \neq \alpha$ $\forall n \in \mathbb{N}$

Since, K is compact, (x_n) has a subsequence (x_{n_k}) s.t.

$(x_{n_k}) \to l \in K$

Observe: $l = \alpha$

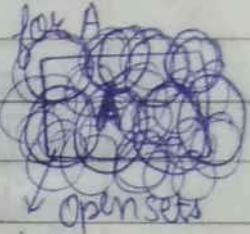
$\therefore \alpha \in K \Rightarrow$ K is closed.

⊛ The Cantor set is compact

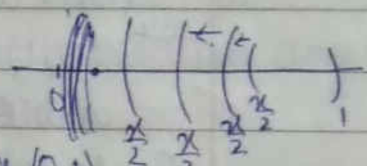
($\because C \subseteq [0,1] \to$ bounded & C is closed)

★ ⊛ $\{O_\lambda : \lambda \in \Lambda\} \to$ Collection of open sets. $A \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$ } Open cover for A

Finite Subcover: Subset of open cover & also A is contained in it.

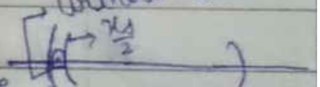


★ $(0,1) \subseteq \bigcup_{x \in (0,1)} \left(\frac{x}{2}, 1\right) \to \left\{\left(\frac{x}{2}, 1\right) : x \in (0,1)\right\}$



So, $\left\{\left(\frac{x}{2}, 1\right) : x \in (0,1)\right\}$ is an open cover for $(0,1)$.

Does this open cover possess any finite subcover?



Let $\left\{\left(\frac{x_1}{2}, 1\right), \left(\frac{x_2}{2}, 1\right), \ldots, \left(\frac{x_m}{2}, 1\right)\right\}$, $m < \infty$ (is finite / be a finite subcover) ~~which is a sub~~

Let $x_s = \min\{x_1, x_2, \ldots, x_m\}$

$\bigcup_{i=1}^{m} \left(\frac{x_i}{2}, 1\right) = \left(\frac{x_s}{2}, 1\right) \not\supseteq (0,1)$ } It cannot cover $(0,1)$.

Observe: $x_s > 0$, so $\frac{x_s}{2} > 0$.

The open cover ~~has~~ $\left\{\left(\frac{x}{2}, 1\right) : x \in (0,1)\right\}$ for $(0,1)$ has no finite subcover

★ $[0,1]$

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$\{(\frac{x}{2},1) : x\in[0,1]\}$ is not an open cover as 0 & 1 are not included in it

So, $\{(\frac{x}{2},1): x\in[0,1]\} \cup \{(-\varepsilon,\varepsilon)\cup(1-\varepsilon,1+\varepsilon)\}$ is open cover for $[0,1]$

Choose $x_0\in(0,1) \ni x_0<\varepsilon$

Cardinality=3 ← $\{(\frac{x_0}{2},1), (-\varepsilon,\varepsilon), (1-\varepsilon, 1+\varepsilon)\}$ is a finite subcover.

② Result: Let $K\subseteq\mathbb{R}$. Then TFAE

① K is compact ② K is closed & bounded

③ Every open cover for K has a finite subcover

E.g: $(0,1)$ has a open cover ⇒ which hasn't finite subcover

So, $(0,1)$ is NOT compact.

{ T.S: $(0,1)$ is NOT compact

$(0,1)$ has a subseq. $(\frac{1}{n+1})$ which won't converge in it.

or $(0,1)$ is bounded but not closed.

or $(0,1)$ has not any open cover which has finite subcover.

So, $(0,1)$ is NOT compact.

★ $\mathbb{Q}$

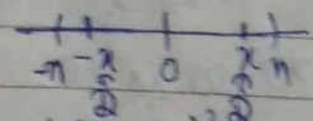
$\pi\in\mathbb{Q}' \Rightarrow \exists (x_n)$ in $\mathbb{Q} \ni (x_n)\to\pi$

No subseq. of (x_n) cgs in $\mathbb{Q}$

$\mathbb{Q}$ is neither closed nor bounded

$\{(-n,n): n\in\mathbb{N}\}$

$\mathbb{Q}\subseteq \bigcup_{n\in\mathbb{N}}(-n,n)$



$\{(-n_1,n_1),(-n_2,n_2),\dots,(-n_s,n_s)\}$, $s<\infty$ is finite but can't cover $\mathbb{Q}$

$n_t=\max\{n_1,\dots,n_s\}$, then $\bigcup_{i=1}^{s}(-n_i,n_i) = (-n_t,n_t)$

★ $\{\frac{1}{n}: n\in\mathbb{N}\}\cup\{0\}$ is compact set

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Q- Check the compactness of the following:

① $\mathbb{Q} \cap [0,1]$ ② $\mathbb{R}$

③ $\left\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\} = A$ ④ $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\} = B$

Solⁿ: ① $\mathbb{Q} \cap [0,1]$ has irrational nos. as its limit point but they don't contained in $\mathbb{Q} \cap [0,1]$, so, $\mathbb{Q} \cap [0,1]$ is not closed and hence, it is not compact.

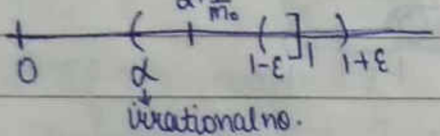
$\frac{\pi}{4}$: limit point of $\mathbb{Q} \cap [0,1]$ $\Rightarrow \exists$ a sq. (x_n) in $\mathbb{Q} \cap [0,1]$ s.t. $(x_n) \to \frac{\pi}{4}$, $x_n \neq \frac{\pi}{4}, n \in \mathbb{N}$

But $\frac{\pi}{4} \notin \mathbb{Q} \cap [0,1]$

$\therefore \mathbb{Q} \cap [0,1]$ is not closed.

$\Downarrow$

$\{x_n\}$ has no subseq. which converges in $\mathbb{Q} \cap [0,1]$.



$\mathbb{Q} \cap [0,1] \subseteq [0,\alpha) \cup (\alpha,1]$

Observe: $\bigcup \left\{\left(\alpha+\frac{1}{n}, 1\right) : n \geq m_0\right\} = (\alpha, 1)$

$\left\{\left(\alpha+\frac{1}{n}, 1\right) : n \geq m_0\right\} \cup (1-\varepsilon, 1+\varepsilon)$ is open cover for $(\alpha, 1]$

$\cup (0,\alpha) \cup (-\varepsilon, \varepsilon)$ is open cover for $[0,\alpha)$

$\to$ open cover which has no finite subcover $\Rightarrow$ Not compact.

② ~~Not clos~~ As $\mathbb{R}$ is not bounded $\Rightarrow$ so it is not compact. (By H.B. Thm)

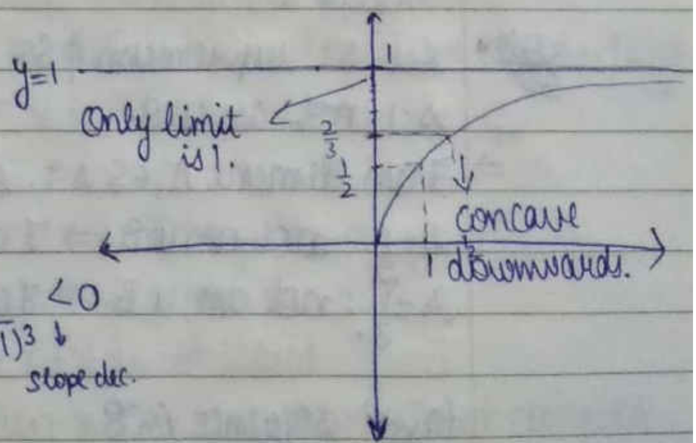
As $\mathbb{R}$ is not bounded, so, by B.W. Thm, it doesn't contain any cgt. subsequence, so, $\mathbb{R}$ is not compact

$\{(-n,n) : n \in \mathbb{N}\}$ is an open cover for $\mathbb{R}$

$\mathbb{R} \subseteq \bigcup_{n \in \mathbb{N}} (-n,n)$

③ $\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$ ~~= A~~

$y = \frac{x}{x+1}, x > 0$



$\frac{dy}{dx} = \frac{1}{(x+1)^2} > 0 \Rightarrow \frac{d^2y}{dx^2} = \frac{-2}{(x+1)^3} < 0$

Inc. funcⁿ slope dec.

$\lim_{x \to \infty} \frac{x}{x+1} = 1$

As 1 is the only limit pt. of A & $1 \in A$, so A is closed and hence it is a compact set.

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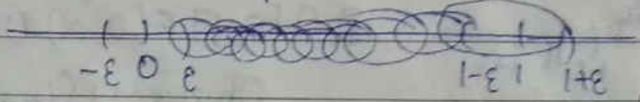
(4) B has the only limit pt. is 0 but $0 \notin B$, so, it is not closed & hence, not compact.
B has no subsequences which converges in B as each subsequence of B converges to 0 but $0 \notin B$, so, B is not compact.

$B \subseteq (1-\varepsilon, 1+\varepsilon) \cup \left(\frac{1}{3}, 1\right) \cup \left(\frac{1}{4}, \frac{1}{2}\right) \cup \left(\frac{1}{5}, \frac{1}{3}\right) \cup \ldots$

Open cover of B which has no finite subcover

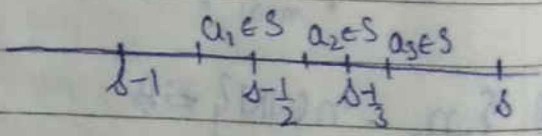
But $B \cup \{0\}$ is compact set



JAM 2015 Let S be a nonempty subset of $\mathbb{R}$. If S is a finite union of disjoint bounded intervals, then which of the following is true?

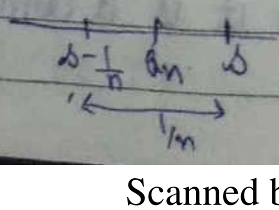
(a) If S is not compact, then $\sup S \notin S$ & $\inf S \notin S$
(b) Even if, $\sup S \in S$ & $\inf S \in S$, S need not be compact.
(c) If $\sup S \in S$ & $\inf S \in S$, then S is compact.
(d) Even if S is compact, it is not necessary that $\sup S \in S$ & $\inf S \in S$

Soln. • Let s: supremum $\notin S$
$s-1$: not an U.B.



$\Rightarrow \exists$ an element $a_1 \in S$ s.t. $s-1 < a_1 < s$
$s-\frac{1}{2}$: not an U.B. $\Rightarrow \exists$ an element $a_2 \in S$ s.t. $s-\frac{1}{2} < a_2 < s$
$s-\frac{1}{3}$: not an U.B. $\Rightarrow \exists$ an element $a_3 \in S$ s.t. $s-\frac{1}{3} < a_3 < s$
⋮

(a_n): sequence in S
Claim: $(a_n) \to s$
given $\varepsilon > 0$
GOAL: $|a_n - s| < \varepsilon \;\; \forall n \geq N$
~~i.e. $s - a_n < \varepsilon$~~



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We have $a_n \in (s-\frac{1}{n}, s) \Rightarrow |a_n - s| < \frac{1}{n} \quad \forall n \in N$

~~N great~~

If we choose N: any natural no. greater than $\frac{1}{\varepsilon}$

$\Rightarrow N > \frac{1}{\varepsilon} \Rightarrow \frac{1}{N} < \varepsilon \Rightarrow \frac{1}{n} < \frac{1}{N} < \varepsilon$ i.e. $n > \frac{1}{\varepsilon}$

$\therefore |a_n - s| < \varepsilon \ \forall n \geq N$

* Result: If A is non-empty bounded above set of $\mathbb{R}$, and $\sup A \notin A$, then $\sup A$ is a limit point of A.
Analogously, for infimum.

Solⁿ: (d) If S is compact
suppose $\sup S \notin S$, then $\sup S \in S'$ —— (*)
But S, being compact, is closed, $\therefore S' \subseteq S$ —— (**)
From (*), (**), $\sup S \in S \Rightarrow\Leftarrow$

* The supremum and infimum of a non empty compact set must be its elements.

Q Show that, if K is compact and F is closed, then $K \cap F$ is compact.
Solⁿ: $K \cap F \subseteq K$, and K is bdd, so, $K \cap F$ is bounded —— ①
K is compact, so, K is closed and F is also closed $\Rightarrow K \cap F$ is closed ②
From ① & ②, we have $K \cap F$ is compact.

CSIR For two subsets X & Y of $\mathbb{R}$, let $X+Y = \{x+y : x \in X, y \in Y\}$
① If X & Y are open sets, then X+Y is open.
② If X & Y are closed sets, then X+Y is closed
③ If X & Y are compact sets, then X+Y is ~~compact closed~~ compact
④ If X is closed, Y is compact, then X+Y is closed.

Solⁿ: ① Let $\alpha \in X+Y$
There exists $\beta \in X$, $\gamma \in Y$ s.t. $\beta + \gamma = \alpha$.
β is an interior pt. of X γ is an interior pt. of Y.

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Ring
$x, s \in R \Rightarrow x+s \in R$
$\Rightarrow x s \in R$
$x \in R \Rightarrow -x \in R$

$0 < x < 1$
$\Downarrow$
$\lim_{n\to\infty} x^n = 0$

$\exists$ an $\varepsilon_1 > 0$ s.t. $(\beta - \varepsilon_1, \beta + \varepsilon_1) \subseteq X$ — (A)

Illy, $\exists$ an $\varepsilon_2 > 0$ s.t. $(\gamma - \varepsilon_2, \gamma + \varepsilon_2) \subseteq Y$ — (B)

(A) + (B) gives, $(\beta+\gamma-(\varepsilon_1+\varepsilon_2), \beta+\gamma+(\varepsilon_1+\varepsilon_2)) \subseteq X+Y$

Observe (Show every pt. of ↑ contains in ↑)

i.e. $(\alpha - (\varepsilon_1+\varepsilon_2), \alpha + (\varepsilon_1+\varepsilon_2))$

$\Rightarrow \alpha \in (X+Y)^\circ$

$\Rightarrow X+Y$ is an Open set.

(2) $\{a+\sqrt{2}b : a, b \in \mathbb{Z}\} \rightarrow$ ~~Ring~~ $= \mathbb{Z}[\sqrt{2}]$ is Euclidean ring.

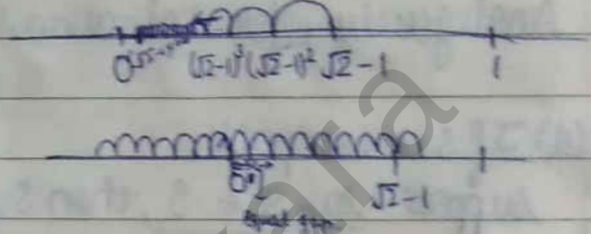
$\sqrt{2}-1 \in \mathbb{Z}[\sqrt{2}]$? Yes

$(0 < \sqrt{2}-1 < 1)$

$(\sqrt{2}-1)^2 \in \mathbb{Z}[\sqrt{2}]$

$0 \leftarrow$ $(\sqrt{2}-1)^3 \in \mathbb{Z}[\sqrt{2}]$

$\vdots$



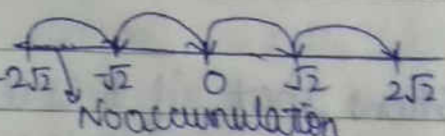
$(\mathbb{Z}[\sqrt{2}])' = \mathbb{R}$

Is $\mathbb{Z}[\sqrt{2}]$ closed? Not b/c $\mathbb{R} \neq \mathbb{Z}[\sqrt{2}]$ (~~every real no. has its limit point inside it~~ (every real no. is limit point of $\mathbb{Z}[\sqrt{2}]$ but that doesn't belong to $\mathbb{Z}\sqrt{2}$)

So, $\mathbb{Z}[\sqrt{2}]$ is not closed.

$\mathbb{Z}[\sqrt{2}] = \mathbb{Z} + \sqrt{2}\mathbb{Z}$

$\mathbb{Z}' = \phi$ & $(\sqrt{2}\mathbb{Z})' = \phi$



So, $\mathbb{Z}$ & $\sqrt{2}\mathbb{Z}$ are closed sets but their sum is not closed.

(4) X: closed set Y: compact set

T.S: $X+Y$ is closed.

$[a \in A' \Leftrightarrow \exists$ a seq. (x_n) in A s.t. $(x_n) \to a, x_n \neq a\ \forall n]$

Let $z \in (X+Y)'$

T.S: $z \in X+Y$

There exists a sequence (z_n) in $X+Y$ s.t. $(z_n) \to z, z_n \neq z\ \forall n$

$z_1 = x_1 + y_1$; $x_1 \in X, y_1 \in Y$

$z_2 = x_2 + y_2$; $x_2 \in X, y_2 \in Y$

$z_3 = x_3 + y_3$, $x_3 \in X$, $y_3 \in Y$

$\vdots$

For each $n \in \mathbb{N}$, $\exists x_n \in X, y_n \in X$ s.t. $z_n = x_n + y_n$

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Let $(y_{n_k}) = (y_{n_1}, y_{n_2}, y_{n_3}, \ldots)$

Let it be $(y_8, y_9, y_{10}, y_{11}, \ldots)$, then

$(x_{n_k}) = (x_8, x_9, x_{10}, x_{11}, \ldots)$, ly, $(z_{n_k}) = (z_8, z_9, z_{10}, z_{11}, \ldots)$

(x_n) : seq. in X

(y_n) : seq. in Y

Y is compact $\Leftrightarrow$ (y_n) has a subsequence (y_{n_k}) which converges to a limit, which is an element of Y.

$(y_{n_k}) \to y \in Y$

$(z_n) \to z \Rightarrow (z_{n_k}) \to z$

$\lim (\underbrace{z_{n_k} - y_{n_k}}_{= x_{n_k}}) = \lim (z_{n_k}) - \lim (y_{n_k}) = z - y$ (By Algebra of Limit)

$\Rightarrow \lim (x_{n_k}) = \underbrace{z - y}_{x, \text{ say}} \Rightarrow x_{n_k}$ is convergent also; $(x_{n_k}) \to x$

(x_{n_k}) : seq. in X. $\Rightarrow x \in X'$

As X is closed, so, $X' \subseteq X \Rightarrow x \in X$

Now, $z - y = x \Rightarrow z = x + y \in X + Y$ ($\because x \in X, y \in Y$)

$\Rightarrow z \in X + Y$

$\therefore X + Y$ is closed.

③ Compact + Closed is closed $\Rightarrow$ Compact + Compact is closed.

X, Y : compact sets

$\Rightarrow X + Y$ is closed —— Ⓐ

X, Y : bounded

[A : bdd $\Rightarrow \exists$ an $M > 0$ s.t. $|x| \leq M \; \forall x \in A$]

T.S: $X + Y$ is bounded

As X is bounded $\Rightarrow \exists M_1 > 0$ s.t. $|x| < M_1, \; \forall x \in X$ —— ①

ly, $\exists M_2 > 0$ s.t. $|y| < M_2 \; \forall y \in Y$ —— ②

① + ② $\Rightarrow |x| + |y| < M_1 + M_2 \quad \forall x \in X, y \in Y$

Also $|x + y| \leq |x| + |y| < M_1 + M_2 \quad \forall x \in X, y \in Y$

$\Rightarrow X + Y$ is bounded —— Ⓑ

Ⓐ & Ⓑ $\Rightarrow X + Y$ is compact.

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$|x_n - \alpha| \le \frac{1}{3^n} < \frac{1}{n} < \epsilon \Rightarrow (x_n) \to \alpha$

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★ $A \subseteq \mathbb{R}$

$x \in A$, $x \notin A'$, then x: isolated point
It must belong to the set.

No pts. of A ($\dot{x}$)

★ $A \subseteq \mathbb{R}$

A: closed set

[$x \in A \Rightarrow$ either x is a limit point of A or an isolated point of A]

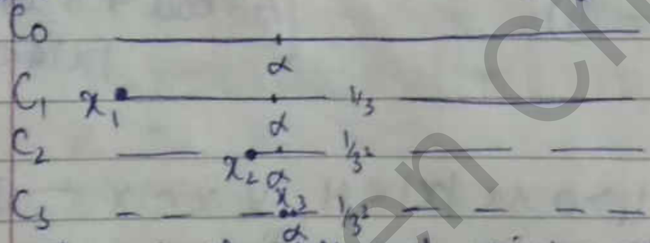
It has no isolated point. ⇒ Each point of A is a limit point of

• Perfect set: A set $A \subseteq \mathbb{R}$ is called a perfect set if it is closed, containing no isolated point.

e.g.: ① $\mathbb{Q}$ is not closed, so, it is not perfect set.
② $[a, b]$ is perfect set
③ Non empty finite sets aren't perfect set
④ Empty set is a perfect set. [$x \in A \Rightarrow x$ has isolated pt.]
⑤ $\mathbb{R}$ is a perfect set
⑥ Cantor set is a perfect set as it has no isolated pt.

$\alpha \in C \to$ Cantor set

$\alpha \in \bigcap_{n \in \mathbb{N}} C_n$



x_n: chosen from C_n. left end point of the component in which α lies

If α is the left end point, we choose the right end point of the component.

(x_n): sequence in C

$\left.\begin{array}{l}(x_n) \to \alpha \\ x_n \ne \alpha \ \forall n\end{array}\right] \Rightarrow \alpha \in C'$

⇒ C has no isolated point, so, it is a perfect set.

★ $\bar{A} = A \cup A'$

When we take closure, the set "expands". (It doesn't expand when A' is empty or A contains all its limit points)

★ A, B: nonempty subsets of $\mathbb{R}$

$\left.A \cap \bar{B} = \phi \ \& \ \bar{A} \cap B = \phi\right\} \Rightarrow$ A & B are separated sets.

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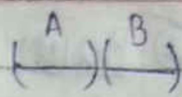
- Separated sets: A, B: non-empty subset of $\mathbb{R}$



$A \cap \bar{B} = \phi$ & $\bar{A} \cap B = \phi$

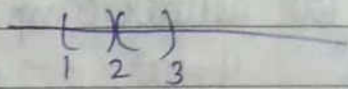
[When A & B are empty then they conditions are true automatically. It is necessary to take both the sets non-empty.]

★ $E \subseteq \mathbb{R}$

If E can be partitioned in two nonempty separated sets, then E is called a disconnected set.

e.g.: ① $(1,2), (2,3) \to$ Separated set.

$E = (1,2) \cup (2,3) \to$ Disconnected set



- A set which is not disconnected, is called a connected set.

e.g.: ② $(5,6], (6,7) \to$ not separated sets.

★ E: connected set

If $E = A \cup B$, then either $A \cap \bar{B} \neq \phi$ or $\bar{A} \cap B \neq \phi$

[$\alpha \in A \cap \bar{B} \Rightarrow \alpha \in A$ & $\alpha \in \bar{B}$

$\alpha \in \bar{B} \Rightarrow \exists (x_n)$ in B s.t. $(x_n) \to \alpha \in A$ [(x_n) convgs. in A]

$\alpha \in \bar{A} \cap B \Rightarrow \exists (y_n)$ in A s.t. $(y_n) \to \alpha \in B$ [(y_n) convgs. in B]]

★ A, B: nonempty separated sets.

$A \cap \bar{B} = \phi$ & $\bar{A} \cap B = \phi$

$B \subseteq \bar{B} \Rightarrow A \cap B \subseteq A \cap \bar{B} = \phi$

$\Rightarrow A \cap B = \phi$

⊛ Separated sets are disjoint.

But converse is not true.

e.g.: $(1,2], (2,3] \to$ Disjoint but not separated

⊛ Disjoint sets may not be separated.

⊛ Result: A set E ($E \neq \phi$) is connected $\Leftrightarrow$ No matter how E is partitioned into two disjoint sets, there exists a sequence in one set which converges converges in the other set.

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$[a,b]$, $a \le b \to$ Singleton set.
$[a,b]$, $a < b$

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Imp

(*) **Result**: $E \neq \emptyset$, $E \subseteq \mathbb{R}$

(Proof) E is connected $\Leftrightarrow$ Either E is a singleton set or an interval.

e.g. $(1,2) \cup (3,4) \to$ Not an interval, hence, not connected

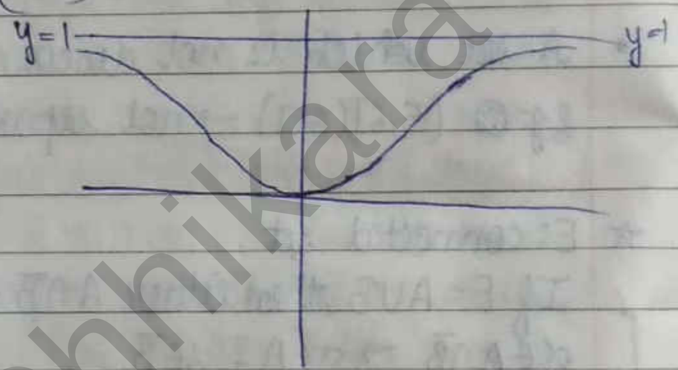
JAM 2014 The set $\left\{ \frac{x^2}{1+x^2} : x \in \mathbb{R} \right\}$ is

(a) connected but not compact in $\mathbb{R}$
(b) compact but not connected in $\mathbb{R}$
(c) compact and connected in $\mathbb{R}$.
(d) neither compact nor connected in $\mathbb{R}$.

Sol^n: $y = \frac{x^2}{1+x^2} \Rightarrow \frac{dy}{dx} = \frac{2x(1+x^2) - x^2(2x)}{(1+x^2)^2}$

even func^n

$= \frac{2x + 2x^3 - 2x^3}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$



$\left.\frac{dy}{dx}\right|_{x=0} = 0$

$\lim_{x\to\infty} \frac{x^2}{1+x^2} = \frac{1}{\frac{1}{x^2}+1} = 1$

$\therefore \left\{ \frac{x^2}{1+x^2} : x \in \mathbb{R} \right\} = [0,1) \to$ Not closed $\Rightarrow$ Not compact

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1/10/16

CSIR Let X be a connected subset of real numbers. If every element of X is irrational, then the cardinality of X is
(a) infinite (b) countably infinite (c) 2 (d) 1

Sol^n: $\alpha, \beta \in X$, $\alpha \neq \beta$
$(\alpha, \beta) \subseteq X$ $\Rightarrow\Leftarrow$ ($\because$ X is an interval)
But (α, β) has rational elements.
So, $|X| = 1$

CSIR Let A be a subset of $\mathbb{R}$ with more than one element. Let $a \in A$. If $A \setminus \{a\}$ is compact, then
① A is compact ② every subset of A is compact
③ A must be a finite set ④ A is disconnected

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X

Solⁿ: Claim: $X' = (X\cup\{\alpha\})'$

i.e. $X\cup\{\alpha\}$ has no "new" "accumative area."

$X \subseteq X\cup\{\alpha\} \Rightarrow X' \subseteq (X\cup\{\alpha\})'$ $(\because A\subseteq B \Rightarrow A'\subseteq B')$

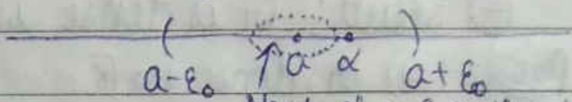
Let $a \in (X\cup\{\alpha\})'$

T.S: $a\in X'$

Let if possible $a\notin X'$

Case I: $a\neq\alpha$

$\exists$ an $\varepsilon_0>0$ s.t. $(a-\varepsilon_0, a+\varepsilon_0)$ containing no point of X other than a



Take $\varepsilon_1 = \frac{1}{2}|a-\alpha|$, $a\neq\alpha$

No pt. of $X\cup\{\alpha\}$ other than a, is there!, $\therefore a$ is not a limit pt. of $X\cup\{\alpha\} \Rightarrow\Leftarrow$

$\therefore a\in X'$

$\Rightarrow (X\cup\{\alpha\})' \subseteq X'$

$\Rightarrow X' = (X\cup\{\alpha\})'$

⊛ If A is a finite set and X is set, then (i) $X' = (X\cup A)'$, (ii) $(X\setminus A)' = X'$

① $A\setminus\{a\}$: compact $\Rightarrow A\setminus\{a\}$ is bounded & closed

$A\setminus\{a\}$ is bounded $\Rightarrow A$ is bounded

$A\setminus\{a\}$ is closed $\Rightarrow (A\setminus\{a\})' \subseteq A\setminus\{a\}$

Also, $A' = (A\setminus\{a\})' \subseteq A\setminus\{a\} \subseteq A \Rightarrow A'\subseteq A \Rightarrow A$ is closed

$\therefore A$ is compact

② $A = [1,2]$

$A\setminus\{1\} = (1,2]$ is not correct example b/c it is given $A\setminus\{a\}$ is compact but $A\setminus\{1\}$ isn't compact and it isn't true for any $a\in A$. $\Rightarrow$ Invalid choice for A

Now, let $A = \{\frac{1}{n} : n\in\mathbb{N}\}\cup\{0\}$ is compact, $a=1$, $A\setminus\{1\}$ is compact $\Big\}\Rightarrow$ Valid choice for A

Not finite

Let $B\subseteq A$, $B = \{\frac{1}{n} : n\in\mathbb{N}\}$

Is B closed? No. b/c $\{0\}\in B'$ but $\{0\}\notin B$

④ As A has more than one element, so, A can't be singleton. Hence, ~~for m~~ to make A connected, A must be of the form $[\alpha,\beta]$, $\underbrace{(\alpha,\beta], [\alpha,\beta), (\alpha,\beta)}$ $\rightarrow$ rejected as A is compact

So, ~~But~~ $A=[\alpha,\beta]$ but for any $a\in A$, $A\setminus\{a\}$ is not compact $\Rightarrow\Leftarrow$

so $[\alpha,\beta]$ is rejected also. $\Rightarrow A$ is not connected.

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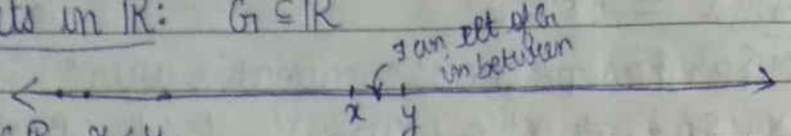
- Dense sets in $\mathbb{R}$: $G \subseteq \mathbb{R}$



Any $x, y \in \mathbb{R}$, $x<y$

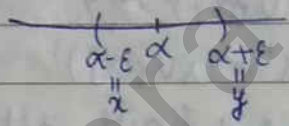
Defⁿ: A set G is said to be dense in $\mathbb{R}$, if given any real numbers x, y ($x<y$), it is possible to find an element $a \in G$ s.t. $x<a<y$

⊛ **Result:** G is dense in $\mathbb{R} \Leftrightarrow \overline{G} = \mathbb{R}$

Proof: ($\Rightarrow$) G: dense in $\mathbb{R}$

Let $\alpha \in \mathbb{R}$

T.S: $\alpha \in \overline{G}$

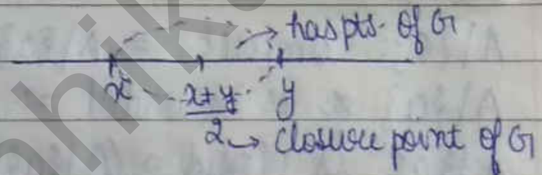


As G is dense, so $\exists$ $a \in G$ s.t.

$x<a<y$

So, $\alpha \in \overline{G}$

($\Leftarrow$) Let $\overline{G} = \mathbb{R}$



$x, y \in \mathbb{R}$, $x<y$

So, G is dense in $\mathbb{R}$.

★ $\mathbb{Q}$ is dense in $\mathbb{R}$ as $\overline{\mathbb{Q}} = \mathbb{R}$

$\mathbb{Z}$ is not dense in $\mathbb{R}$ as $\overline{\mathbb{Z}} \neq \mathbb{R}$

- Nowhere dense sets: A set $E \subseteq \mathbb{R}$ is said to be nowhere dense, if $\overline{E}$ has no open interval.

⊛ **Result:** E is nowhere dense $\overset{①}{\Leftrightarrow}$ $(\overline{E})^{\circ} \overset{②}{=} \emptyset \Leftrightarrow (\overline{E})^c$ is dense in $\mathbb{R}$. ③

"The closure has empty interior"

Proof: ① ⇒ ② Let E is nowhere dense

T.S: $(\overline{E})^{\circ} = \emptyset$

$\alpha \in \mathbb{R}$

T.S: $\alpha \notin (\overline{E})^{\circ}$

Let if possible, $\alpha \in (\overline{E})^{\circ}$

$\exists$ an $\varepsilon_0 > 0$ s.t. $(\alpha - \varepsilon_0, \alpha + \varepsilon_0) \subseteq \overline{E}$ →← ($\because$ E is nowhere dense)

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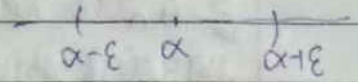
② ⇒ ① (Ex.)

① ⇒ ③ E: nowhere dense

I.S: $(\overline{E})^c$ is dense in $\mathbb{R}$

Let $\alpha \in \mathbb{R}$

T.S: α is a closure point of $(\overline{E})^c$



$(\alpha-\varepsilon, \alpha+\varepsilon) \nsubseteq \overline{E}$ as E is nowhere dense

$\Rightarrow \exists\, a \in (\alpha-\varepsilon, \alpha+\varepsilon)$ s.t. $a \in (\overline{E})^c \Rightarrow (\alpha-\varepsilon, \alpha+\varepsilon) \subseteq (\overline{E})^c$

③ ⇒ ① (Ex.)

Q: Decide whether the following sets are dense in $\mathbb{R}$, nowhere dense, or somewhere in between

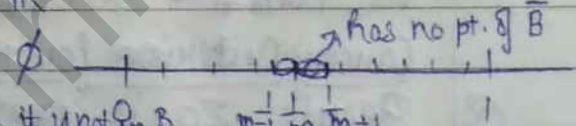
(a) $A = \mathbb{Q} \cap [0,5]$ (b) $B = \{\frac{1}{n} : n \in \mathbb{N}\}$

(c) $\mathbb{R} \setminus \mathbb{Q}$ (d) the Cantor Set C.

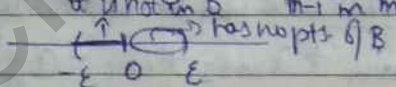
Solⁿ (a) $(\overline{A})^\circ = [0,5]^\circ = (0,5) \neq \emptyset \Rightarrow$ it is not nowhere dense in $\mathbb{R}$

$\overline{A} \neq \mathbb{R}$, ∴ it is not dense in $\mathbb{R}$

(b) $(\overline{B})^\circ = \left(\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}\right)^\circ = \emptyset$



$(-\varepsilon, \varepsilon) \nsubseteq \overline{B}$ for any $\varepsilon > 0$



$\Rightarrow 0 \notin (\overline{B})^\circ$

$\frac{1}{m} \in (\overline{B})^\circ$? No as $\left(\frac{1}{m} - \varepsilon, \frac{1}{m} + \varepsilon\right) \nsubseteq \overline{B} \Rightarrow \frac{1}{m} \notin (\overline{B})^\circ$

Also, $1 \notin (\overline{B})^\circ$

∴ $(\overline{B})^\circ = \emptyset \Rightarrow$ B is not dense but nowhere dense.

(c) $\overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R} \Rightarrow \mathbb{R} \setminus \mathbb{Q}$ is ~~At~~ dense in $\mathbb{R}$

$(\overline{\mathbb{R} \setminus \mathbb{Q}})^\circ = \mathbb{R} \Rightarrow \mathbb{R} \setminus \mathbb{Q}$ is not nowhere dense in $\mathbb{R}$.

(d) $\overline{C} = C \Rightarrow C$ is not dense in $\mathbb{R}$.

$(\overline{C})^\circ$ C has no open interval i.e. $\overline{C}$ has no open interval.

Hence, C is nowhere dense. (By defⁿ)

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Series

★ (a_n): sequence

$a_1 + a_2 + a_3 + \ldots$

$\sum_{n=1}^{\infty} a_n$

$s_1 = a_1$
$s_2 = a_1 + a_2$
$s_3 = a_1 + a_2 + a_3$
$\vdots$

sequence of partial sums

$\sum a_n$ is said to be convergent if its sequence (s_n) of partial sums is convergent.

★ $\sum \frac{1}{n}$ is divergent and $\sum \frac{1}{n^2}$ is convergent.

Result: $\sum \frac{1}{n^p}$, $p>0$ is convergent if $p>1$
divergent if $0<p\leq 1$

$\sum a_n$: convergent $\Leftrightarrow$ (s_n) is convergent $\Leftrightarrow$ (s_n) is Cauchy
$\quad$ $\downarrow$ sequence of partial sums

$\Leftrightarrow$ for an $\varepsilon>0$, $\exists$ an $N \in \mathbb{N}$ s.t. $|s_n - s_m| < \varepsilon \ \forall n, m \geq N \ (n>m)$

i.e., $|(a_1 + a_2 + \ldots + a_m + a_{m+1} + \ldots + a_n) - (a_1 + \ldots + a_m)| < \varepsilon \ \forall n>m\geq N$

i.e., $|a_{m+1} + \ldots + a_n| < \varepsilon \ \forall n>m\geq N$

Cauchy criterion for the convergence of series:

Result: $\sum a_n$ is convergent $\Leftrightarrow$ for each $\varepsilon>0$, $\exists$ an $N \in \mathbb{N}$ s.t.
$|a_{m+1} + a_{m+2} + \ldots + a_n| < \varepsilon \ \forall n>m\geq N$

★ $\sum a_n$ is convergent $\Rightarrow$ lim s_n exists

~~$\lim(s_{n+1} - s_n) = \lim(s_{n+1}) - \lim(s_n)$~~

$\lim(\underbrace{s_n - s_{n-1}}_{a_n}) = \lim(s_n) - \lim(s_{n-1}) = 0$

$\Rightarrow \lim a_n = 0$

Result: $\sum a_n$ is convergent $\Rightarrow \lim a_n = 0$ $\rightarrow$ Necessary condition for the convergence of a series

Converse? Not true

e.g: $\sum \frac{1}{n}$: divergent
but $\lim_{n\to\infty} \frac{1}{n} = 0$

It is necessary but not sufficient.

Q: Discuss the convergence of

① $\sum \left(\frac{1}{n}\right)^{1/n}$ $\quad$ ② $\sum \cos\frac{1}{n^2}$

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$P \Rightarrow Q \rightarrow$ necessary condition

$P \Leftarrow Q \rightarrow$ Sufficient condition

$P \Rightarrow Q$ or $\cancel{Q} \Rightarrow \cancel{P}$

Solⁿ: ① $Y = \lim_{n\to\infty} \frac{1}{n^{1/n}}$

$\log Y = \log \lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} \log n^{1/n} = \lim_{n\to\infty} \frac{\log n}{n}$

$= \lim_{n\to\infty} \frac{1/n}{1} = 0$

$\Rightarrow \lim \frac{1}{n^{1/n}} = 1 \neq 0 \Rightarrow \sum \left(\frac{1}{n}\right)^{1/n}$ is divergent

② $\lim_{n\to\infty} \cos \frac{1}{n^2} = \cos \lim_{n\to\infty} \frac{1}{n^2}$ ($\because$ cos is cont. so, it commute with limit)

$= \cos 0 = 1 \neq 0$

$\Rightarrow \cancel{\lim} \sum \cos \frac{1}{n^2}$ is divergent.

⊛ If $\lim a_n \neq 0$, then $\sum a_n$ cannot convergent.

V.Imp. NBHM 2012

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Pick out the cases which imply that the sequence is Cauchy

(a) $|x_n - x_{n+1}| \le \frac{1}{n} \ \forall n$ ✓(b) $|x_n - x_{n+1}| \le \frac{1}{n^2} \ \forall n$

✓(c) $|x_n - x_{n+1}| \le \frac{1}{2^n} \ \forall n$

Solⁿ: (a) Take $x_n = \sum_{k=1}^{n} \frac{1}{k}$

(x_n): sequence of partial sums of $\sum \frac{1}{k}$ is not cgt. $\Rightarrow$ not Cauchy

$|x_n - x_{n+1}| = \frac{1}{n+1} \le \frac{1}{n}$

⋆ Pseudo-Cauchy Sequence: A sequence (a_n) is said to be pseudo-cauchy sequence if for each $\varepsilon > 0$, $\exists$ an $N \in \mathbb{N}$ s.t. $|a_{n+1} - a_n| < \varepsilon \ \forall n \ge N$

⊛ A pseudo-cauchy sequence may not converge.

it looks to be happen but may or may not happen

(b) $|x_n - x_m| = |x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} - x_{n-3} + \dots + x_{m+1} - x_m|$

$\le |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + |x_{n-2} - x_{n-3}| + \dots + |x_{m+1} - x_m|$

$\le \frac{1}{(n-1)^2} + \frac{1}{(n-2)^2} + \dots + \frac{1}{m^2}$ ——— ①

$\sum \frac{1}{n^2}$ is convergent $\Rightarrow$ for a given $\varepsilon > 0$, $\exists$ an $N \in \mathbb{N}$ s.t. [By Cauchy's Criterion]

$\left| \frac{1}{(m+1)^2} + \frac{1}{(m+2)^2} + \dots + \frac{1}{n^2} \right| < \varepsilon \ \forall m > n \ge N$

i.e. $\frac{1}{(m+1)^2} + \frac{1}{(m+2)^2} + \dots + \frac{1}{n^2} < \varepsilon$ ⊛

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$\therefore$ ① $< \frac{1}{n^2} + \frac{1}{n^2} \cdots + \frac{1}{m^2}$, if $m > n \geq N$

$\therefore |x_n - x_m| < \varepsilon$

So, it is cauchy

(c) $\sum \frac{1}{2^n}$ this is convergent, so, $\{x_n\}_{n=1}^{\infty}$ is cauchy.

⊛ Suppose $|x_{n+1} - x_n| \leq a_n \; \forall n$

Then, ① If $\sum a_n$ is ~~convergent~~ Cauchy, then (x_n) is ~~convergent~~ Cauchy.

② If $\sum a_n$ is not Cauchy, then (x_n) is not cauchy.

JAM 2010 Which of the following conditions doesnot ensure the convergence of a real sequence (a_n)?

(a) $|a_n - a_{n+1}| \to 0$ as $n \to \infty$ (b) $\sum |a_n - a_{n+1}|$ is convergent

(c) $\sum_{n=1}^{\infty} n a_n$ is convergent (d) The sequences $\{a_{2n}\}$, $\{a_{2n+1}\}$ & $\{a_{3n}\}$ are convergent

Sol^n: (a) $a_n = \sum_{k=1}^{n} \frac{1}{k}$ but $\lim a_n$ doesn't exist

$\Rightarrow (a_n)$ is not convergent.

(c) $\sum n a_n$ is cgt. $\Rightarrow \lim n a_n = 0$

$\lim a_n$ exists, if not so, then $\lim n a_n \neq 0$

and must be zero, if not so, then $\lim n a_n = \infty$ or $-\infty$

$\therefore \lim a_n = 0$

(d) $\{a_{2n}\} \to l_1$ $\{a_{2n+1}\} \to l_2$ $\{a_{3n}\} \to l_3$

$\{a_{6n}\} = \{a_6, a_{12}, \ldots\}$ is a subsequence of $\{a_{2n}\}$ as well as $\{a_{3n}\}$

$\Rightarrow l_1 = l_3$ $(l_1 = l_3 \Leftarrow \{a_{6n}\} \xrightarrow{u} l_1$ & $\{a_{6n}\} \xrightarrow{u} l_3)$

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$a_n \geq 0 \Rightarrow \lim a_n \geq 0$

So $\frac{u_n}{v_n} > 0 \Rightarrow l > 0$

2/10/16

- Comparison Tests:

Imp

$\Sigma u_n, \Sigma v_n$: ⊕ve term series

1) There exists an $N \in \mathbb{N}$ s.t. $u_n \leq v_n \;\forall\, n \geq N$

2) Σv_n is convergent

Then Σu_n is also convergent.

Q: Test the convergence of

① $1 + \frac{1}{2^2} + \frac{1}{3^3} + \dots + \frac{1}{n^n}$ ② $\Sigma \frac{1}{\sqrt{n!}}$ ③ $\Sigma \frac{1}{n^2 \log n}$

Solⁿ ① $1 + \frac{1}{2^2} + \frac{1}{3^3} + \dots + \frac{1}{n^n} + \dots \leq 1 + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} + \dots$ is convergent.

② $\frac{1}{\sqrt{n!}} = \frac{1}{\sqrt{1.2.3\dots n}} \leq \frac{1}{\sqrt{2^{n-1}}} = \frac{1}{2^{(n-1)/2}} = 1 + \frac{1}{2^{1/2}} + \frac{1}{2^1} + \frac{1}{2^{3/2}} + \dots$

$1 + \frac{1}{2^{1/2}} + \frac{1}{2^1} + \dots + \frac{1}{2^{(n-1)/2}}$ is a Geometric series with ratio $\frac{1}{2^{1/2}}$

So, it is convergent & hence $\Sigma \frac{1}{\sqrt{n!}}$ converges.

③ $\Sigma \frac{1}{n^2 \log n} \leq \Sigma \frac{1}{n^2}$ and $\Sigma \frac{1}{n^2}$ cgs, so, $\Sigma \frac{1}{n^2 \log n}$ cgs.

⊛ Result: $\Sigma u_n, \Sigma v_n$: ⊕ve term series

Suppose $\lim \frac{u_n}{v_n} = l$, where l is neither zero nor infinite, $l > 0$

Then Σu_n & Σv_n have the same behaviour in relation to their convergence.

Proof: $l > 0$, set $\epsilon = l/2$

There exists an $N \in \mathbb{N}$ s.t. $\left|\frac{u_n}{v_n} - l\right| < \frac{l}{2} \;\forall\, n \geq N$

$\Rightarrow \frac{l}{2} < \frac{u_n}{v_n} < \frac{3l}{2} \;\forall\, n \geq N$

$\Rightarrow \frac{l}{2} v_n < u_n < \frac{3l}{2} v_n$

If Σv_n is cgt, then $\Sigma \frac{3l}{2} v_n$ is cgt and by above test, u_n is cgt.

If Σv_n is dgt, then $\Sigma \frac{l}{2} v_n$ is dgt and hence, u_n is dgt

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Q: Examine the convergence of

① $\sum \frac{1}{n^2+a^2}$ ② $\sum \frac{1}{\sqrt{n}+\sqrt{n+1}}$ ③ $\sum \frac{bn-a}{bn^2+a^2}$

④ $\sum \left[\sqrt{n^4+1}-\sqrt{n^4-1}\right]$ ⑤ $\sum \sin\frac{1}{n}$

Solⁿ: ① $u_n = \frac{1}{n^2+a^2}$, $v_n = \frac{1}{n^2}$

$\lim \frac{u_n}{v_n} = \lim \frac{n^2}{n^2+a^2} = 1 > 0$

As v_n is cgt, so, u_n is also cgt.

② $u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}}$, $v_n = \frac{1}{\sqrt{n}+\sqrt{n}} = \frac{1}{2\sqrt{n}}$

$\lim \frac{u_n}{v_n} = \frac{2\sqrt{n}}{\sqrt{n}+\sqrt{n+1}} = 1 > 0$

As v_n is dgt., so, u_n is also dgt.

③ $u_n = \frac{bn-a}{bn^2+a^2}$, $v_n = \frac{n}{n^2} = \frac{1}{n}$

As v_n dgs, so, u_n dgs.

④ $u_n = \sqrt{n^4+1}-\sqrt{n^4-1}$ $v_n = \frac{1}{\sqrt{n^4-1}+\sqrt{n^4+1}} = \frac{1}{2n^2}$

As v_n cgt, so, u_n is cgt.

⑤ $u_n = \sin\frac{1}{n}$, $v_n = \frac{1}{n}$

$\lim \frac{u_n}{v_n} = \frac{\sin 1/n}{1/n} = 1$

So $\sum u_n$ dgs, as, v_n dgs.

★ 1^∞ case: $\lim_{x\to a} f(x) = 1$ $\lim_{x\to a} g(x) = \infty$

$\lim_{x\to a} \{f(x)\}^{g(x)} = e^{\lim_{x\to a} (f(x)-1)\cdot g(x)}$

• D'Alembert's ratio test:

$\sum u_n$: (+)ve term series

Let $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l$

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Then if (i) $l > 1$, Σu_n converges
(ii) $l < 1$, Σu_n diverges
(iii) $l = 1$, then the test fails!

Further, if l is infinite, then Σu_n converges

Q- Test the convergence

① $\Sigma \frac{(n+1)!}{3^n}$ ② $\Sigma \frac{2^{n-1}}{3^n+1}$ ③ $\Sigma \frac{1^2 \cdot 2^2}{1!} + \frac{2^2 \cdot 3^2}{2!} + \dots + \dots$

Solⁿ: ① $u_n = \frac{(n+1)!}{3^n}$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)!}{3^n} \times \frac{3^{n+1}}{(n+2)!} = \frac{3}{n+2} \rightarrow 0 < 1$$

So Σu_n is divergent.

② $u_n = \frac{2^{n-1}}{3^n+1}$

$$\frac{u_n}{u_{n+1}} = \frac{2^{n-1}}{3^n+1} \times \frac{3^{n+1}+1}{2^n} = \frac{1}{2} \frac{3^{n+1}+1}{3^n+1} = \frac{1}{2}\left[\frac{3+3^{-n}}{1+3^{-n}}\right] \rightarrow \frac{3}{2} > 1$$

So, Σu_n is convergent

③ $u_n = \frac{n^2 \cdot (n+1)^2}{n!}$

$$\frac{u_n}{u_{n+1}} = \frac{n^2 \cdot \cancel{(n+1)^2}}{n!} \times \frac{(n+1)!}{\cancel{(n+1)^2}(n+2)^2} = \frac{n^2(n+1)}{(n+2)^2} \rightarrow \infty > 1$$

So, Σu_n is convergent.

NBHM 2012 Pick out the convergent series

① $\Sigma [(n^3+1)^{1/3} - n]$ ② $\Sigma \frac{(n+1)^n}{n^{n+3/2}}$ ③ $\Sigma \frac{1}{n^{1+\frac{1}{n}}}$

Solⁿ: $u_n = (n^3+1)^{1/3} - (n^3)^{1/3}$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

Let $(n^3+1)^{1/3} = a$ $(n^3)^{1/3} = b$

$$\Rightarrow u_n = \frac{(n^3+1) - n^3}{(n^3+1)^{2/3} + (n^3)^{2/3} + (n^3+1)^{1/3}(n^3)^{1/3}} = \frac{1}{(n^3+1)^{2/3} + n^2 + (n^3+1)^{1/3} \cdot n}$$

Let $v_n = \frac{1}{(n^3)^{2/3} + n^2 + (n^3)^{1/3} \cdot n} = \frac{1}{n^2+n^2+n^2} = \frac{1}{3n^2}$ is cgt.

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In A.P.
$a_n = a+(n-1)d$

So, Σu_n is cgt.

(2) $u_n = \frac{(n+1)^n}{n^{n+3/2}} = \left(\frac{n+1}{n}\right)^n \cdot \frac{1}{n^{3/2}}$

Let $v_n = \frac{1}{n^{3/2}}$

$\lim \frac{u_n}{v_n} = \lim \left(\frac{n+1}{n}\right)^n = \lim \left(1+\frac{1}{n}\right)^n = e > 0$

As v_n is cgt so, u_n is also cgt.

(3) $\Sigma u_n = \frac{1}{n^{1+\frac{1}{n}}} = \frac{1}{n \cdot n^{1/n}}$

Let $v_n = \frac{1}{n}$

$\lim \frac{u_n}{v_n} = \frac{1}{n^{1/n}} = 1 > 0$

$\left[y = \lim n^{1/n} \Rightarrow \log y = \lim \frac{1}{n} \log n = 0 \right.$
$\left. \Rightarrow y = e^0 = 1 \right]$

As v_n is dgt, so, u_n is also dgt.

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$\frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{3\cdot4\cdot5} + \frac{7}{4\cdot5\cdot6} + \cdots$

$u_n = \frac{2n-1}{n(n+1)(n+2)}$

$v_n = \frac{n}{n\cdot n\cdot n} = \frac{1}{n^2}$

$\lim \frac{u_n}{v_n} \Rightarrow$ is finite and non-zero

Σv_n cgs, so, Σu_n also cgs.

TIFR

T/F. The series $\Sigma \frac{\sqrt{n+1}-\sqrt{n}}{n}$ diverges.

Sol^n: $u_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$ $\quad v_n = \frac{1}{2n^{3/2}}$

As v_n is cgt., so, u_n is cgt.

- Cauchy's n^{th} root test: Σu_n: +ve term series

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Suppose $\lim_{n\to\infty} [u_n]^{1/n} = l$

If ① $l<1$, then Σu_n converges

② $l>1$, then Σu_n diverges

③ $l=1$, then test fails!

Q- Test the convergence of:

① $\Sigma \left(1+\frac{1}{n}\right)^{-n^2}$ ② $\Sigma \frac{n^{n^2}}{(n+1)^{n^2}}$ ③ $\Sigma \left(\frac{nx}{n+1}\right)^n, x>0$

Sol^n: ① $[u_n]^{1/n} = \left(1+\frac{1}{n}\right)^{-n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \to \frac{1}{e} < 1$

So, Σu_n is cgt.

② $\Sigma [u_n]^{1/n} = \left(\frac{n^{n^2}}{(n+1)^{n^2}}\right)^{1/n} = \frac{n^n}{(1+n)^n} = \left(\frac{n}{1+n}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n} \to \frac{1}{e} < 1$

So, Σu_n is cgt.

③ $[u_n]^{1/n} = \frac{nx}{n+1} \to x$

If $x>1$, Σu_n diverges
$0<x<1$, Σu_n converges] ~~(By Cauchy's nth root Test)~~

For $x=1$, $u_n = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n} \to \frac{1}{e} \neq 0$ ($\because \Sigma u_n$ cgt $\Rightarrow \lim u_n = 0$)

$\Rightarrow$ For $x=1$, Σu_n is dgt.

So, If $x \geq 1$ then Σu_n divergent
$0<x<1$, then Σu_n convergent.

- **Cauchy's Integral Test:**

~~Σu_n: positive term series~~

$u(x)$: non-negative function, monotonically decreasing

$u(n) = u_n$

Then $\sum_{n=1}^{\infty} u_n$ is convergent $\Leftrightarrow \int_1^{\infty} u(x)\,dx$ is convergent.

Finite value

e.g: $\Sigma u_n = \Sigma \frac{1}{n^2}$

Take $u(x) = \frac{1}{x^2}$

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$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left[\frac{-1}{x}\right]_1^{t}$$

$$= \lim_{t\to\infty} -\frac{1}{t} - (-1) = 1$$

$\Rightarrow \int_1^\infty u(x)\,dx$ is cgt $\Rightarrow \sum_{n=1}^{\infty} u_n$ is cgt.

Imp Q: $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$, $p>0$

Solⁿ: For $p=1$

$$u_n = \frac{1}{n\log n}$$

$$u(x) = \frac{1}{x\log x}$$

$$\int_2^\infty u(x)\,dx = \lim_{t\to\infty}\int_2^{t} \frac{1}{x\log x}\,dx = \lim_{t\to\infty}\int_2^t \frac{1/x}{\log x}\,dx$$

$$= \lim_{t\to\infty}\left[\log|\log x|\right]_2^t = \lim_{t\to\infty} \log|\log t| - \log|\log 2| = \infty$$

So, $\int_2^\infty u(x)\,dx$ is not cgt $\Rightarrow$ For $p=1$, $\sum u_n$ is not cgt.

For $p>1$,

$$\int_2^\infty \frac{1}{x(\log x)^p}\,dx = \int_2^\infty \frac{1/x}{(\log x)^p}\,dx = \lim_{t\to\infty}\int_2^t \frac{1/x}{(\log x)^p}\,dx$$

$$= \lim_{t\to\infty}\left[\frac{(\log x)^{-p+1}}{-p+1}\right]_2^t = \lim_{t\to\infty} \frac{(\log t)^{-p+1}}{-p+1} - \frac{(\log 2)^{-p+1}}{-p+1}$$

is finite as $p>1 \Rightarrow -p+1<0 \Rightarrow$ Cgt.

For $0<p<1$, $\Rightarrow -p+1>0$, then $\int_2^\infty \frac{1}{x(\log x)^p}\,dx \to \infty$, so dgt.

If $p>1$, convergent

If $0<p\le 1$, divergent

- Leibnitz's test: (u_n) : ⊕ve term sequence

① $u_1 \geq u_2 \geq u_3 \geq \dots$

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② $\lim_{n\to\infty} u_n = 0$

Then the alternating series $\sum (-1)^n u_n$ is convergent.

e.g.: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \longrightarrow \log 2$

$u_n = \frac{1}{n}$ is monotonically decreasing

and $\lim_{n\to\infty} u_n = 0$

$\therefore \sum (-1)^n u_n = \sum \frac{(-1)^n}{n}$ is convergent.

- $\sum u_n$: series

$\sum |u_n|$: convergent

$s_n = u_1 + u_2 + \ldots + u_n$

$t_n = |u_1| + |u_2| + \ldots + |u_n|$

(t_n) is convergent, as $\sum |u_n|$ is convergent.

Is (s_n) convergent? Yes

$|s_n - s_m| = |u_{m+1} + \ldots + u_n| \le |u_{m+1}| + \ldots + |u_n| \to |t_n - t_m| < \varepsilon$ (as t_n is cgt.)

$\therefore \sum u_n$ is cgt.

⊛ Result: If $\sum |u_n|$ is convergent, then $\sum u_n$ is convergent.

converse? It is not true

e.g. $\sum \frac{1}{n}$ diverges but $\sum \left\{ \frac{(-1)^n}{n} \right.$ converges

$\left| \frac{(-1)^n}{n} \right|$

- If $\sum |u_n|$ converges, $\sum u_n$: Absolutely convergent series

If $\sum u_n$ converges, but $\sum |u_n|$ does not converge.

$\sum u_n$: conditionally convergent series

e.g. $\sum \frac{(-1)^n}{n}$ is conditionally convergent series.

$\sum \frac{1}{n^2}$ is absolutely convergent series

⊛ Every +ve term series which is convergent, is absolutely convergent series

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$\log x < x \Rightarrow \log n < n \Rightarrow \frac{1}{\log n} > \frac{1}{n}$

8/10/16 By Comparison Test, $\frac{1}{\log n}$ is dgt.

Q- Test the convergence, absolute convergence and conditional convergence of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\log(n+1)}$

Solⁿ: Leibnitz's Test

(a_n): +ve term, monotonically dec., $(a_n) \to 0$ $\Bigg\} \Rightarrow \sum (-1)^n a_n$ converges

~~Here~~ Here, $a_n = \frac{1}{\log(n+1)}$ → +ve terms & mon. dec.

$(a_n) \to 0$

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\log(n+1)}$ converges.

* $\sum |a_n|$ converges $\Rightarrow \sum a_n$ converges. Converse is not true.

If $\sum a_n$ cgs, then $\sum |a_n|$ may not converge.

$\sum \left|\frac{(-1)^{n+1}}{\log(n+1)}\right| = \sum \frac{1}{\log(n+1)}$

$u_n = \frac{1}{\log(n+1)}$

★ * $u(x)$: MD, +ve.
$u(n) = u_n \ \forall n$
$\int_2^{\infty} \frac{1}{\log x}\, dx = ?$
Cauchy Integral Test

If $\frac{1}{\log(n+1)}$ converges, then (a_n) is A.C. and if not, then (a_n) is C.C. ~~B~~ As by comp. test $\sum \frac{1}{\log(n+1)}$ dgs, so, (a_n) is C.C.

Q- Show that the series

$\frac{\log 2}{2^2} - \frac{\log 3}{3^2} + \frac{\log 4}{4^2} - \dots$ converges.

Solⁿ: $\sum_{n=2}^{\infty} \frac{\log n}{n^2}$ & $u_n = \frac{\log n}{n^2}$

$u(x) := \frac{\log x}{x^2}$; $x > 0$ $\left[\frac{MD}{+ve}\right]$

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$$u'(x) = \frac{x - 2x(\log x)}{x^4} = \frac{[1 - 2\log x]}{x^3} < 0 \quad \text{if } x > e^{1/2}.$$

$\left[\text{if } 1 - 2\log x < 0 \Rightarrow 1 < 2\log x \Rightarrow \frac{1}{2} < \log x \Rightarrow x > e^{1/2}\right]$

$$\lim_{n\to\infty} \frac{\log n}{n^2} = \lim_{n\to\infty} \frac{1/n}{2n} = \lim_{n\to\infty} \frac{1}{2n^2} = 0$$

$\therefore \sum_{n=2}^{\infty} (-1)^n u_n$ is cgt.

CSIR Which of the following is convergent?

① $\sum \frac{1}{\sqrt{n+1} - \sqrt{n}}$ ② $\sum \frac{\sin n}{n}$ ✓

③ $\sum (-1)^n \log n$ ④ $\sum \frac{\log n}{n}$

Solⁿ: ① $u_n = \frac{1}{\sqrt{n+1} - \sqrt{n}} \times \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \sqrt{n+1} + \sqrt{n}$

It is not cgt. as.

$\sum u_n$ cgs $\Rightarrow \lim u_n = 0$ but $\lim [\sqrt{n+1} + \sqrt{n}] = \infty$.

③ $\lim_{n\to\infty} (-1)^n \log n \neq 0$

$\Rightarrow$ It is not convergent.

④ $u(x) = \frac{\log x}{x}, x > 0$

$$u'(x) = \frac{1 - \log x}{x^2} < 0, \text{ if } x > e$$

$$\int_1^{\infty} \frac{\log x}{x} dx$$

Put $\log x = t \Rightarrow \frac{1}{x} dx = dt$

$$= \left[\frac{(\log x)^2}{2}\right]_0^{\infty} = \infty$$

$\Rightarrow$ It is also convergent.

- Dirichlet's Test: If (u_n): +ve term sq.
  $(u_n) \longrightarrow 0$
  $\sum a_n$: series with sq. of partial sums is bounded,

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then $\sum a_n u_n$ is convergent.

(2) $a_n = \sin n$, $u_n = \frac{1}{n}$

Is $\sum \sin n$ bounded? Yes

$s_n = \sin 1 + \sin 2 + \dots + \sin n$

$|s_n| = |\sin 1 + \sin 2 + \dots + \sin n|$

$\leq |\sin 1| + \dots + |\sin n|$

$\not\leq n$

$\Rightarrow \sum \sin n$ is bounded

- Abel's Test: If (u_n): +ve term sequence bounded
$\sum a_n$: whose sq. of Partial sum is cgt.
then $\sum a_n u_n$ is cgt.

CSIR Let $\{a_n\}$ and $\{b_n\}$ be sequence of real numbers satisfying $|a_n| \leq |b_n|$ for all $n \geq 1$, then

(1) $\sum a_n$ converges whenever $\sum b_n$ converges.

(2) $\sum a_n$ converges absolutely, whenever $\sum b_n$ converges absolutely

(3) $\sum b_n$ converges whenever $\sum a_n$ converges

(4) $\sum b_n$ converges absolutely whenever $\sum a_n$ converges absolutely

Solⁿ: (1) Let $\sum b_n = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ is cgt. by Leibnitz's Test.
$\frac{(-1)^{n+1}}{n}$ and $\sum a_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ is not cgt.
Also $|a_n| = |b_n|$

(2) By Comparison test., $\sum |a_n|$ converges when $\sum |b_n|$ converges
as $|a_n| \leq |b_n|$

Also, $|a_n|, |b_n|$: +ve term series.

(3) No, Let $\sum a_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$
& $\sum b_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots$ [as $|a_n| = |b_n|$]

(4) Given : $\sum |a_n|$ convergent.
Is $\sum |b_n|$ convergent? No
Let $a_n = \frac{1}{n^2}$, $b_n = \frac{1}{n}$
$\sum \left|\frac{1}{n^2}\right| = \sum \frac{1}{n^2}$ is cgt. but $\sum \left|\frac{1}{n}\right|$ is not cgt.

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CSIR

If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then which of the following is NOT true?

① $\sum_{m=n}^{\infty} a_m \to 0$ as $n \to \infty$ (True) ② $\sum_{n=1}^{\infty} a_n \sin n$ is convergent. (True)

③ $\sum_{n=1}^{\infty} e^{a_n}$ is convergent. ④ $\sum_{n=1}^{\infty} a_n^2$ is divergent.

given: $\sum_{n=1}^{\infty} a_n$ is A.C. $\Rightarrow \sum a_n$ cgt; $s_n = a_1 + a_2 + \dots + a_m$

sol^n: ① $|s_m - s_n| = |a_{m+1} + a_{m+2} + \dots + a_m| < \varepsilon \;\forall m, n \geq N, m \geq n$

$|\sum_{k=m+1}^{m} a_k| < \varepsilon \;\forall m, n \geq N \Rightarrow m \geq n$

Letting $m \to \infty$, we get

$\underbrace{|\sum_{k=m+1}^{\infty} a_k|}_{= b_{m+1}} \leq \varepsilon \;\forall m \geq N$

If $a_n < K \;\forall n$
$\Rightarrow \lim a_n \leq K$

$(b_{m+1}) \to 0$

(s_n) is cgt. $\Rightarrow (s_n)$ is cauchy.
given $\varepsilon > 0 \;\forall N \in \mathbb{N}$ s.t.

$b_{n+1} = \sum_{k=n+1}^{\infty} a_k$

$|b_{n+1}| \leq \varepsilon \;\forall n \geq N$

i.e. $|b_{n+1} - 0| \leq \varepsilon \;\forall n \geq N$

$\Rightarrow (b_{n+1}) \to 0$ or $(b_n) \to 0$

, where $b_n = \sum_{k=n}^{\infty} a_k \to 0$ as $n \to \infty$

② $|a_n \sin n| \leq |a_n|$

As $\sum |a_n|$ is cgt. $\Rightarrow \sum |a_n \sin n|$ is cgt.

③ $\lim_{n\to\infty} e^{a_n} = e^{\lim a_n} = e^0 = 1$ ($\because \sum a_n$ is cgt.) $\neq 0$
($\because e$ is cont. so, commute with limit)

$\therefore$ Divergent

④ $a_n = \frac{1}{n^2}$ & $a_n^2 = \frac{1}{n^4}$ cgt.

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Power Series

Power Series $\begin{cases} a_0 + a_1 x + a_2 x^2 + \cdots \\ \sum a_n x^n \end{cases}$

a_n: coefficients, independent of x.

Q- ~~$\sum a_n$~~ For what values of x, does the power series $\sum a_n x^n$ converge?

(*) Result: If the power series $\sum a_n x^n$ converges for $x = \alpha$, then the power series converges absolutely for $x = \beta$, where $|\beta| < |\alpha|$

Proof: Given: $\sum a_n \alpha^n$ converges.

Then $\lim_{n\to\infty} a_n \alpha^n = 0$

Let $\varepsilon = \frac{1}{2}$, Then $\exists$ an $N \in \mathbb{N}$ s.t. $|a_n \alpha^n - 0| < \frac{1}{2}$ $\forall n \geq N$

i.e., $|a_n \alpha^n| < \frac{1}{2}$ $\forall n \geq N$

Check $\sum a_n \beta^n$

$|a_n \beta^n| = \left|a_n \left(\frac{\beta}{\alpha}\right)^n \cdot \alpha^n\right| < \frac{1}{2}\left(\frac{\beta}{\alpha}\right)^n$ $\forall n \geq N$

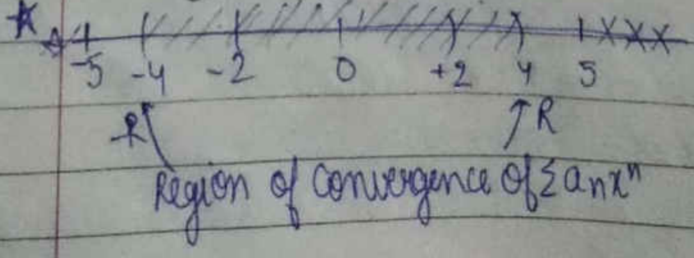
$\sum \frac{1}{2}\left|\frac{\beta}{\alpha}\right|^n$ is G.S. with const. ratio $\left|\frac{\beta}{\alpha}\right| < 1$

As $|\beta| < |\alpha| \Rightarrow \frac{|\beta|}{|\alpha|} < 1$ i.e. $|\beta/\alpha| < 1$

$\therefore \sum |a_n \beta^n|$ is convergent i.e. $\sum a_n \beta^n$ is abs. convergent.

(*) Result: If the ~~pot~~ power series $\sum a_n x^n$ diverges for $x = \gamma$, then the power series diverges for $x = \delta$, where $|\delta| < |\gamma|$

JAM If a power series $\sum a_n x^n$ converges for $x = 3$, then the series
(a) converges absolutely for $x = -2$. ✓
(b) converges but not absolutely for $x = -1$.
(c) converges but not absolutely for $x = 1$
(d) ~~converges~~ diverges for $x = -2$.



$[-R, R]$
$[-R, R)$
$(-R, R]$
$(-R, R)$

Only possible formats of region of convergence of $\sum a_n x^n$

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R: radius of convergence.

CSIR Let $\{a_n : n \geq 1\}$ be a sequence of real numbers s.t. $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} |a_n|$ is divergent. Let R be the radius of convergence of the power series $\sum a_n x^n$, then we can conclude that

(a) $0 < R < 1$ ✓(b) $R = 1$ (c) $1 < R < \infty$ (d) $R = \infty$

Sol^n: For $x=1$, $\sum a_n x^n$ converges. $\left(\because \sum_{n=1}^{\infty} a_n \text{ is cgt.}\right)$

$(-1,1) \rightarrow \sum a_n x^n$ is abs. cgt.

$\Rightarrow R \geq 1$

If ~~For~~ $|\alpha| > 1$, check the convergence of $\sum a_n \alpha^n$

If $\sum a_n \alpha^n$ converges, then the series $\sum a_n$ is cgs absolutely. ~~⇒~~

We got it, by putting $x=1$.

$\therefore$ For $|\alpha| > 1$, $\sum a_n \alpha^n$ is not convergent.

$\Rightarrow R = 1$

✱ Result: $\sum a_n x^n$ power series

If $\overline{\lim} |a_n|^{1/n} = \frac{1}{R}$, then R.o.c is R.

✱ If (a_n) cgs, then $\overline{\lim}\, a_n = \underline{\lim}\, a_n = l$

Largest limit pt ← Limit superior Limit Inferior → Smallest limit pt.

✱ If $\overline{\lim} |a_n|^{1/n} = 0$, then we write $R = \infty$ → "Everywhere Convergent"

If $\overline{\lim} |a_n|^{1/n} = \infty$, then we write $R = 0$ → "Nowhere Convergent" (cgt only for zero)

Q- ① ~~The~~ $1 + 2x + 3x^2 + \cdots$

② $\sum \frac{x^{n-1}}{n^2}$

③ $\sum (-1)^n x^n$

④ $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Sol^n ① $1 + 2x + 3x^2 + \cdots = \sum n x^{n-1} = \frac{1}{x} \sum n x^n$

$a_n = n$

$\overline{\lim} |a_n|^{1/n} = \overline{\lim}\, n^{1/n}$ $\left[n^{1/n} \rightarrow 1 \text{ as } n \rightarrow \infty \right]$

$= 1 = \frac{1}{R}$

$\Rightarrow R = 1$

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Interval of Convergence:

Suspicious pts: For $x=1$

Σn

$(a_n) \to \infty$, when $n \to \infty$

$\therefore \Sigma n$ is dgt.

For $x=-1$

$\Sigma (-1)^n n$

(a_n) is dgt.

$\therefore \Sigma (-1)^n n$ is dgt.

$\therefore$ IOC $= (-1, 1)$

② $\Sigma \frac{x^{n-1}}{n^2} = \frac{1}{x} \Sigma \frac{x^n}{n^2}$

$a_n = \frac{1}{n^2}$

$\overline{\lim} |a_n|^{1/n} = \left(\frac{1}{n^2}\right)^{1/n} = 1 = \frac{1}{R}$

$\left[\text{Let } y = \left(\frac{1}{n^2}\right)^{1/n} \right.$

$\log y = \frac{1}{n} \log \frac{1}{n^2} = \frac{\log 1/n^2}{n} = \frac{\cancel{1/x^2}\; n^2 \times -1/n^3}{1}$

$= \frac{-n^2}{n^3} = \frac{-1}{n}$

$\lim_{n\to\infty} \log y = \lim_{n\to\infty} \frac{-1}{n} = 0$

$\Rightarrow \log \lim_{n\to\infty} y = 0 \Rightarrow \lim_{n\to\infty} y = e^0 = 1 \Big]$

$\Rightarrow R = 1$

Interval of Convergence:

Suspicious pts: For $x=1$

$\Sigma \frac{1}{n^2}$ is cgt.

$\therefore$ IOC $= [-1, 1]$

For $x=-1$

$\Sigma \frac{(-1)^n}{n^2}$ is cgt. by Leibnitz's test

③ $\Sigma (-1)^n x^n$

$a_n = (-1)^n$

$\overline{\lim} |a_n|^{1/n} = \overline{\lim} |(-1)^n|^{1/n} = 1 = \frac{1}{R}$

$\Rightarrow R = 1$

For $x=1$, $\Sigma (-1)^n$ is dgt.

For $x=-1$, $\Sigma (-1)^n (-1)^n = \Sigma (-1)^{2n} = \Sigma 1$ is dgt.

$\therefore$ IOC $= (-1, 1)$

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(4) $1+x+\frac{x^2}{2!}+\dots = \sum \frac{x^n}{n!}$

$a_n = \frac{1}{n!}$

$\overline{\lim} |a_n|^{1/n} = \overline{\lim} \left(\frac{1}{n!}\right)^{1/n} = \overline{\lim} \frac{1}{(n!)^{1/n}} = \frac{1}{e} = \frac{1}{R}$

[* Cauchy's first theorem on limits.
Cauchy's second theorem on limits $\Rightarrow \lim (n!)^{1/n} = e$]

$\Rightarrow R = e$ — Direct Application of Cauchy's 2nd Thm.

~~For $x = e$,~~

Q: The power series $\sum \frac{[2+(-1)^n]^n}{3^n} x^n$ converges

(1) only for $x=0$ (2) only for $-1<x<1$

(3) for all $x \in \mathbb{R}$ (4) only for $-1<x\leq 1$

Solⁿ: $a_n = \frac{[2+(-1)^n]^n}{3^n}$

$\overline{\lim} |a_n|^{1/n} = \overline{\lim} \left|\frac{2+(-1)^n}{3}\right| = \frac{1}{3}, 1, \frac{1}{3}, 1, \dots$

Limit pts. of $\frac{2+(-1)^n}{3}$ are $\frac{1}{3}, 1$

$\therefore \overline{\lim} |a_n|^{1/n} = 1 = \frac{1}{R}$

$\Rightarrow R = 1$

For $x=1$, $\sum \frac{[2+(-1)^n]^n}{3^n}$ $\left(\frac{2+(-1)^n}{3}\right)^n \neq 0$

$\frac{1}{3}, 1, \frac{1}{3^2}, 1, \frac{1}{3^3}, 1, \frac{1}{3^4}, \dots$

$\therefore$ For $x=1$, $\sum a_n x^n$ is not cgt.

JAM 2010 The radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^{n^2}$, where $a_0 = 1$, $a_n = 3^{-n} a_{n-1}$ for all $n \in \mathbb{N}$, is

(a) 0 (b) $\sqrt{3}$ (c) 3 (d) ∞

$a_n = \frac{1}{3^n} a_{n-1} = \frac{1}{3^n} \cdot \frac{1}{3^{n-1}} a_{n-2} = \frac{1}{3^n} \cdot \frac{1}{3^{n-1}} \cdot \frac{1}{3^{n-2}} \cdots \frac{1}{3} \cdot 1$

$= \frac{1}{3^{1+2+\dots+n}} = \frac{1}{3^{\frac{n(n+1)}{2}}}$

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$n^2 \to$ Cauchy n^{th} root test

Linear, $2n+5 \to$ De-Alembert Ratio Test

$\lim |a_n|^{1/n} = \lim \left| \frac{1}{3^{(n+1)/2}} . z^n \right|$

For the convergence of Σa_n, $\lim \left| \frac{1}{3^{(n+1)/2}} z^n \right| < 1$

i.e. $\lim \left| \frac{z^n}{(\sqrt{3})^n . \sqrt{3}} \right| < 1$ i.e. $\lim \left| \left(\frac{z}{\sqrt{3}}\right)^n \right| < \sqrt{3}$

[Σr^n cgs iff $-1 \le r < 1$]

This i.e. $\lim \left| \frac{z}{\sqrt{3}} \right|^n < \sqrt{3}$

This limit exist only when $\left| \frac{z}{\sqrt{3}} \right| < 1$ i.e. $|z| < |\sqrt{3}|$
(and it is $0 < \sqrt{3}$)

$\Rightarrow R = \sqrt{3}$

JAM 2009 Let $a_n = \begin{cases} 1/3^n & \text{, if } n \text{ is prime} \\ 1/4^n & \text{, if } n \text{ is not prime} \end{cases}$

The radius of convergence of $\Sigma a_n x^n$ is

(a) 4 ✓(b) 3 (c) $\frac{1}{3}$ (d) $\frac{1}{4}$

Sol^n: If $a_n = \frac{1}{3^n}$, then $\overline{\lim} \left| \frac{1}{3^n} \right|^{1/n} = \overline{\lim} \frac{1}{3} = \frac{1}{3} = \frac{1}{R_1} \Rightarrow R_1 = 3$

If $a_n = \frac{1}{4^n}$, then $\overline{\lim} \left| \frac{1}{4^n} \right|^{1/n} = \frac{1}{4} = \frac{1}{R_2} \Rightarrow R_2 = 4$

$R = \min\{R_1, R_2\} = 3$

Q: If $a_n = \begin{cases} 1/3^n, & \text{if } n = 3m \\ 1/4^n & \text{if } n = 3m+1 \\ 1/5^n & \text{if } n = 3m+2 \end{cases}$, then $R = 3$

If we take $R=4$, then prime terms diverge. (-4 -3 3 4)

JAM 2007 Suppose (C_n) is a seq. of real numbers s.t. $\lim |C_n|^{1/n}$ exists & is non zero. If the radius of convergence of $\Sigma C_n x^n$ is equal to r, then the radius of convergence of $\Sigma n^2 C_n x^n$ is

(a) less than r (b) greater than r

✓(c) equal to r (d) equal to 0.

Sol^n: $\lim |C_n|^{1/n} = \frac{1}{r}$

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$\lim |n^2 c_n|^{1/n} = \lim (n^{1/n})^2 \cdot |c_n| = 1 \cdot \frac{1}{R} = \frac{1}{R}$.

(2) Result: $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ for $\sum a_n x^n$ derived from De 'Alembert Root Test.

Q: (1) $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \longrightarrow \sum \frac{x^n}{n!}$

(2) $\frac{1}{2} \cdot x + \frac{1}{2} \cdot \frac{3}{5} x^2 + \frac{1}{2} \cdot \frac{3}{5} \cdot \frac{5}{8} x^3 + \dots \rightarrow \sum \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 5 \cdot 8 \cdots (3n-1)} x^n$

Solⁿ: (1) $a_n = \frac{1}{n!}$

$R = \lim \frac{1/n!}{1/(n+1)!} = \lim n+1 = \infty$

$\therefore \sum a_n x^n$ is cgt. everywhere.

(2) $a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$

$R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{3(n+1)-1}{2(n+1)-1} = \lim_{n\to\infty} \frac{3n+2}{2n+1} = \frac{3}{2}$

JAM 2015: If the power series $\sum \frac{n!}{n^n} x^{2n}$ converges for $|x|<c$ and diverges for $|x|>c$, then $c = ?$

Solⁿ: let $x^2 = y$, then $\sum \frac{n!}{n^n} x^{2n} = \sum \frac{n!}{n^n} y^n$

$R = \lim \frac{(n+1)^{n+1}}{(n+1) \cdot n^n}$

$= \lim \frac{(n+1)^n}{n^n} = \lim \left(\frac{1}{(1+1/n)^n} \right) = e$

$|y| < e \overset{\text{i.e.}}{\Rightarrow} |x^2| < e$ i.e. $|x| < \sqrt{e} \rightarrow c = \sqrt{e}$.

Q: The set of all points x at which $\sum \frac{n}{(2n+1)^2} (x-2)^{3n}$ converges is

Solⁿ: $\sum \frac{n}{(2n+1)^2} y^n$, $y = (x-2)^3$.

$R = \lim \left(\frac{n}{n+1}\right) \frac{(2n+3)^2}{(2n+1)^2} = 1$

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$|y| < 1 \Rightarrow |(x-2)^3| < 1 \Rightarrow |x-2|^3 < 1$

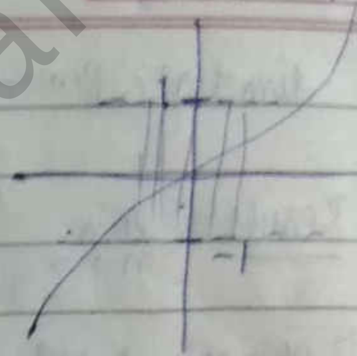
$\Rightarrow |x-2| < 1$

$-1 < x-2 < 1$

$1 < x < 3$

For $x = 1$

For $x = 3$



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